STOCHASTIC CHARACTERISTICS OF WAVE GROUPS IN RANDOM SEAS

BY

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TO
MY
PARENTS

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This paper presents the results of a study to develop a method for predicting various stochastic characteristics of wave groups in random seas where the water depth is sufficiently deep. Wave groups are defined in the present study as a sequence of at least two high waves having nearly equal periods. Formulae to statistically predict, for a given wave spectrum, (i) time duration of a wave group, (ii) number of waves in a group, and (iii) frequency of occurrence of wave groups in a specified time are newly derived. The probability density functions of time duration of wave group as well as time interval between successive wave groups are analytically developed based on the concept of a narrow-banded Gaussian random process. Renewal theory is applied to the probability density function of time interval between wave groups for evaluating the frequency of occurrence of wave groups in a specified time. The results of numerical computations carried out using wave data

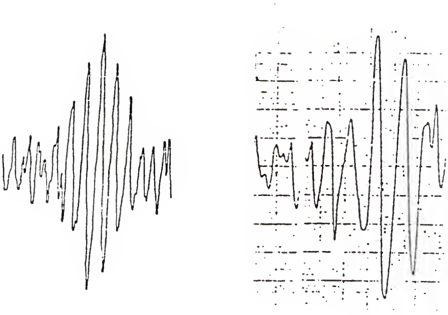
measured in the North Sea show that the average time duration of a wave group computed by the present theory agrees reasonably well with the observed data. Also, the frequency of occurrence of wave groups predicted by the present theory agrees well with those observed, while the frequency of occurrence computed by applying the formula currently used is substantially greater than observed data.

CHAPTER I
INTRODUCTION

A unique phenomenon associated with the wind generated waves in the ocean is a sequence of high waves having nearly equal periods commonly known as wave groups. Two examples of wave groups observed in the ocean are shown in Figure 1.1. Figure 1.1(a) is taken from Rye (1974) in which he shows group waves recorded in the North Sea during a storm of significant wave height of 10.0 meters, while Figure 1.1(b) shows wave groups recorded in severe seas by weather ship at Station K in North Atlantic.

It has been known that wave groups often cause problems for safety of marine systems when the period of the individual waves in the group is closed to the marine system's natural motion period. This is not because wave heights are exceptionally large but occurs primarily because of motion augmentation due to the resonance with the waves which, in turn, can induce capsizing. As another example, a moored marine system tends to respond to successive high waves which induce a slow drift oscillation of the system resulting in large forces on the mooring lines.

Since the wave group phenomenon is extremely interesting and important for the consideration of marine systems, many studies have been carried out on the various phases of the phenomenon. One area of considerable interest is the stochastic analysis of wave groups in random seas.



(a) (b)

Figure 1.1 Examples of wave groups observed at sea (Example (a) from Rye (1974)).

Several studies have been carried out, to date, on the stochastic analysis of wave groups. However, in almost all these studies, the length of time that the wave group persists ($\tau_{\alpha+}$ in Figure 1.2), the number of waves involved in a group, and the time interval between successive occurrences of wave groups (τ_{α} in Figure 1.2) are all average values and no precise probabilistic information on these individual quantities has been provided.

Furthermore, extreme care must be taken in applying the currently available methods for evaluating the mean number of waves and mean time of duration of wave groups due to the following reason. It has been customary, to date, to consider the exceedance of the envelope above a certain level to identify a wave group. However, this is not correct; if the time duration $\tau_{\alpha+}$ shown in Figure 1.2 is relatively short, there may be one wave crest (or no wave crest) in time $\tau_{\alpha+}$, which obviously does not constitute a wave group even though the envelope exceeds the specified level. In order to elaborate this statement, Figure 1.3 shows a pictorial sketch of the time interval between successive upcrossings of the envelope of a random process. Suppose the upcrossing of the envelope at point B in the Figure 1.3 is not associated with wave groups, then the time interval AC (instead of AB and BC) should be considered as time interval between wave groups.

It is the purpose of this study to develop an analytical method for evaluating various stochastic characteristics of wave groups in random seas from a given wave spectrum. The water depth is considered to be sufficiently deep in the present study. The analytical method includes the evaluation of

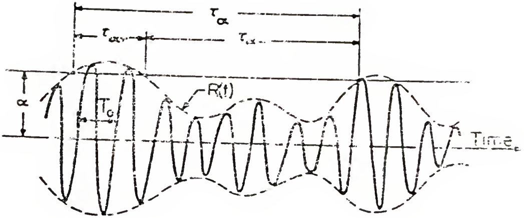


Figure 1.2 Level crossing of the envelope of a random process.

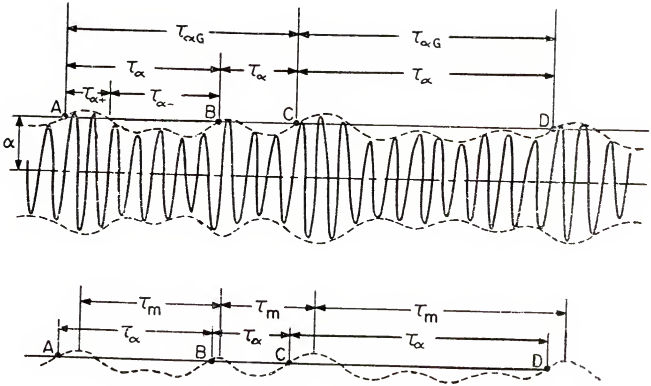


Figure 1.3 Definition of time interval between successive wave groups, $\tau_{\alpha G}$.

(a) The probability density function of the time duration of wave group,

(b) The probability of occurrence of a specified number of waves in a group,

(c) The probability density function of the time interval between successive wave groups, and

(d) Frequency of occurrence of wave groups in a given sea.

In the derivation of the probability density functions of this study, the wave group phenomenon is treated as a level crossing problem associated with the envelope of a narrow band Gaussian random process (R(t) in Figure 1.2), and then the probability density functions are modified taking the condition required for the wave groups into consideration. Wave groups are defined in the present study as a sequence of at least two high waves exceeding a specified level. For evaluating the frequency of occurrence of wave groups in a specified time, renewal theory is applied to the probability density function of the time interval between wave groups.

The average time duration associated with wave groups computed by the newly developed theory and the predicted frequency of occurrence of wave groups agree well with the data observed in the North Sea.

This paper consists of three chapters followed by conclusions and eleven appendices. Chapter II reviews the currently available literature on the stochastic characteristics of wave groups in random seas where the water depth is sufficiently deep.

In Chapter III, the time duration of wave group and the number of waves in a wave group are considered. The probability density function applicable for the length of time that the wave group persists is

analytically developed. Formula to evaluate the probability of the number of waves in a group is derived based on the probability density function of time duration. The average time duration associated with wave groups computed by the newly developed theory was compared with the data observed in the North Sea.

In Chapter IV, a method to estimate the frequency of occurrence of wave groups is discussed. The probability density function of time intervals between successive wave groups is derived and then a method to evaluate the frequency of occurrence of wave groups in a specified time is developed by applying renewal theory. Measured data in the North Sea and the predicted frequency of occurrence of wave groups are compared.

In Chapter V, the findings and conclusions obtained from the results of the present study are summarized.

In the Appendices, additional derivations to clarify or to prove formulae used in the main text are presented.

CHAPTER II
REVIEW OF THE LITERATURE

This chapter reviews the currently available literature on the stochastic characteristics of wave groups in random seas.

Two different approaches have been applied to the stochastic analysis of the wave group phenomenon. One approach is to consider wave groups as a level crossing problem associated with the envelope of random processes. The other approach regards the sequence of wave heights as a Markov chain with a correlation only between two successive waves.

The first approach was used by Rice (1945, 1958) in his pioneering work on the mathematical analysis of a Gaussian noise and was first applied to the stochastic analysis of wave groups in random seas by Longuet-Higgins (1957).

Longuet-Higgins (1957, 1984) derived formulae to evaluate the mean length of the run (i.e., the mean number of waves included in a wave group), the mean time interval between successive upcrossings of a specified level by an envelope, and the mean time duration of a wave group, in terms of spectral moments and the band width parameter of the frequency spectrum. Assuming that the upcrossings by the wave envelope of a certain level were related with wave groups and followed a Poisson random process, the intervals between successive wave groups were shown to be exponentially distributed. Furthermore, he stated that the two approaches (wave envelope approach and the Markov chain approach) led to almost equivalent results and both could be applied only to sufficiently

narrow band random processes, and to data in which higher and lower frequencies were filtered out.

The mean number of waves included in a run of high waves and the average number of waves between successive occurrences of wave groups were given by asymptotic formulae derived by Ewing (1973). The frequency spectrum was assumed to be symmetric about its mean frequency and narrow banded. Also presented were the asymptotic formulae as functions of the spectral peakedness parameter, which represented the degree of concentration of the frequency spectrum about its mean frequency. When the spectral peakedness parameter had large values (very narrow banded frequency spectra), the predicted mean values by Ewing's asymptotic formulae agreed well with data from numerical wave simulation experiments. However, Chakrabarti et al. (1974) commented that the mean length of the run of high waves between successive occurrences of wave groups was smaller than the observed values in wave data from a North Atlantic storm.

Nolte and Hsu (1972) studied the average time duration of the wave envelope exceedances over a specified level (denoted by $\tau_{\alpha+}$ in Figure 1.2), the average frequency of occurrences of the upcrossings by a wave envelope of a high level, and the average time between successive level upcrossings by an envelope ($\overline{\tau}_{\alpha}$ in Figure 1.2). The wave envelope upcrossings of a certain level were considered to be related with wave groups. By assuming that the wave group occurrences were a Poisson random process, a formula was derived to evaluate the probability of a specified number of wave groups in a certain time interval.

Goda (1976) assumed that the successive wave heights were statistically independent. By applying the theory of runs (Fisz, 1963), he

calculated the probability that a sequence of wave heights higher than a certain level (i.e., a wave group) contained a specified number of wave heights. Using the envelope theory, the mean time duration of the envelope persistence above a specified level was derived. Also developed was a formula for evaluating the mean length of the run of high waves as a function of the spectral peakedness parameter. The predicted mean number of waves in a wave group was compared with that which was observed in field wave data and numerical wave simulation data, and he concluded that the degree of the spectral peakedness influenced the mean number of waves in a wave group. However, Wilson and Baird (1972) found that the mean value of the number of waves included in a wave group evaluated by using Goda's formula was smaller than the observed value in wave data obtained in the ocean off Nova Scotia.

Rye (1979, 1981) found that the degree of the spectral peakedness was a significant factor for the wave group formation and noted that the correlation coefficient of two consecutive wave heights was strongly related with the spectral peakedness parameter. Furthermore, Rye (1974) observed in wave data from the North Sea, that the wave group formation appeared to be more pronounced during the growing stage of wind generated seas than during the decaying stage.

Mase et al. (1983) calculated the mean length of wave groups (i.e., the mean number of waves in a wave group) and the mean number of waves between consecutive wave groups by using numerical wave simulation and field data. Both average values from the two types of data were compared in order to study the degree of wave grouping.

Elgar et al. (1984) argued that the mean length of wave groups could be generally related with the spectral peakedness parameter. They calculated values of the mean length of wave groups from numerical wave simulation data, plotted them against the spectral peakedness parameter, and found that the spectral peakedness parameter and the mean length of the wave groups are well correlated for broad banded frequency spectra.

The second approach that regards a sequence of wave heights as a Markov chain was used first by Kimura (1980). The correlation coefficient of consecutive waves was evaluated from numerical wave simulation data, and thereby estimates of the mean number of waves in a wave group and the mean number of waves between successive wave groups were obtained.

Battjes and Vledder (1984) estimated the correlation coefficient of two consecutive wave heights by using a spectral shape parameter based on the theory of Gaussian processes. Their results for the mean length of the run and for the mean number of waves between successive wave groups agreed with those derived by Kimura.

Arhan and Ezraty (1978) developed a formula for the joint probability density function of two consecutive wave heights that can be used to calculate, by the means of Markov chain theory, the average number of waves in a wave group.

As this brief review indicates, several studies have been conducted for the stochastic characteristics of wave groups in random seas. However, in almost all these studies, the number of waves included in a wave group and the time duration of a wave group over a certain level are average values, and no precise probabilistic formulae for these

individual quantities have been provided. The analysis of wave records has indicated that the currently available theories underpredicted the mean length and the mean time duration of wave groups. Furthermore, most of the studies on wave groups assumed that the envelope upcrossings of a specific level were related with wave groups and followed a Poisson random process. However, it is highly desirable to verify the validity of this assumption for evaluating the frequency of occurrence of wave groups in random seas.

CHAPTER III
TIME DURATION OF AND NUMBER OF WAVES IN A WAVE GROUP

3.1 Probability Distribution of Time Duration of Envelope Exceeding a Certain Level

As stated in the Introduction, an exceedance of the envelope above a specified level α does not necessarily mean a wave group unless two or more wave crests are compromised in the exceedance. Therefore, we first derive the probability density function of the time duration, $\tau_{\alpha+}$ associated with the envelope exceeding a level α; then this probability density function will be truncated taking into consideration the condition required for the existence of a wave group.

In the development of the probability density function of time duration of the envelope exceeding a specified level α, waves are assumed to be a Gaussian random process with a narrow-band sprectrum. Then, following Rice's approach (1945), the probability that the envelope of a random process x(t) exceeds a specified level α at time t_1 with velocity $\dot{R}_1$ during an upward crossing and then crosses that same level downward at time t_2 with velocity $\dot{R}_2$ is given approximately by the following formula:

$$\Pr\left\{\begin{array}{l}\text{Upward crossing of a level } \alpha \text{ at time } t_1\\ \text{with velocity } \dot{R}_1 \text{ followed by a downward}\\ \text{crossing at time } t_2 \text{ with velocity } \dot{R}_2\end{array}\right\} = \frac{\bar{N}_{\alpha+}\ (\tau_{\alpha+})}{\bar{N}_{\alpha+}}$$

$$= \frac{\displaystyle\int_{-\infty}^{0}\int_{0}^{\infty} \dot{R}_1\dot{R}_2 \; f(\alpha,\dot{R}_1,\alpha,\dot{R}_2)\; d\dot{R}_1 \; d\dot{R}_2}{\displaystyle\int_{0}^{\infty} \dot{R}_1 \; f(\alpha,\dot{R}_1)\; d\dot{R}_1} \qquad \begin{array}{l} o \leqslant \dot{R}_1 < \infty \\ -\infty < \dot{R}_2 \leqslant o \end{array} \quad (3.1)$$

Here, the numerator, $\bar{N}_{\alpha+}(\tau_{\alpha+})$, represents the expected number (per unit time) of envelope upward crossings with velocity $\dot{R}_1$ followed by the downward crossings with velocity $\dot{R}_2$, while the denominator, $\bar{N}_{\alpha+}$, represents the expected number (per unit time) of envelope upward crossings of the level α with velocity $\dot{R}_1$. The function $f(\alpha, \dot{R}_1, \alpha, \dot{R}_2)$ is the joint probability density function of displacement R and velocity $\dot{R}$ at times t_1 and t_2 with $R_1 = R_2 = \alpha$ in this case, while $f(\alpha, \dot{R}_1)$ is the joint probability density of displacement R and velocity $\dot{R}$ of the envelope at time t_1 with $R_1 = \alpha$.

The probability given in Eq. (3.1) may be interpreted as the time that the envelope spends above the level α. In other words, by letting $t_1 - t_2 = \tau_{\alpha+}$, Eq. (3.1) is equivalent to the probability density function of $\tau_{\alpha+}$ for a specified level α.

Integration of the joint probability density function involved in the numerator of Eq. (3.1) cannot be carried out directly. It is necessary to first obtain the joint probability density function of the wave profile x(t) and velocity $\dot{x}(t)$ at times t_1 and t_2, and then to transform this joint probability density function of displacement and velocity to that of the wave envelope and its velocity.

Assuming a narrow-band Gaussian random process, the wave profile x(t) can be written as

$$x(t) = \sum_{n=1}^{\infty} c_n \cos(\omega_n t + \varepsilon_n) = x_c(t) \cos \bar{\omega}\, t - x_s(t) \sin \bar{\omega}\, t \tag{3.2}$$

where

$$x_c(t) = \sum_{n=1}^{\infty} c_n \cos \{(\omega_n - \bar{\omega})t - \varepsilon_n\}$$

(3.3)

$$x_s(t) = \sum_{n=1}^{\infty} c_n \sin \{(\omega_n - \bar{\omega})t - \varepsilon_n\}$$

$\bar{\omega}$ = mean frequency of wave spectrum

Let the random variable X_{c1} represent the value of $x_c\,(t)$ at time t_1, and the random variable X_{s1} represent the value of $x_s(t)$ at time t_1, etc. Since we take sine and cosine components for both displacement and velocity at times t_1 and t_2, we have a set of eight random variables $(X_{c1}, \dot{X}_{c1}, X_{c2}, \dot{X}_{c2}, X_{s1}, \dot{X}_{s1}, X_{s2}, \dot{X}_{s2})$ which composes a random vector $\mathbf{X}$. Here, each element of $\mathbf{X}$ obeys the normal probability distribution, and we may write the joint normal probability density function in the following form:

$$f(\mathbf{X}) = \frac{1}{(2\,\pi)^4} \cdot \frac{1}{|\Sigma|^{1/2}}\, e^{-\frac{1}{2}\mathbf{X}'\Sigma^{-1}\mathbf{X}} \tag{3.4}$$

where $\mathbf{X}' = (x_{c1}, \dot{x}_{c1}, x_{c2}, \dot{x}_{c2}, x_{s1}, \dot{x}_{s1}, x_{s2}, \dot{x}_{s2})$

and Σ is the covariance matrix given by

$$\Sigma = \begin{pmatrix} \mu_o & 0 & \nu_o & -\nu_1 & 0 & 0 & \eta_o & \eta_1 \\ 0 & \mu_2 & \nu_1 & \nu_2 & 0 & 0 & -\eta_1 & \eta_2 \\ \nu_o & \nu_1 & \mu_o & 0 & -\eta_o & \eta_1 & 0 & 0 \\ -\nu_1 & \nu_2 & 0 & \mu_2 & -\eta_1 & -\eta_2 & 0 & 0 \\ 0 & 0 & -\eta_o & -\eta_1 & \mu_o & 0 & \nu_o & -\nu_1 \\ 0 & 0 & \eta_1 & -\eta_2 & 0 & \mu_2 & \nu_1 & \nu_2 \\ \eta_o & -\eta_1 & 0 & 0 & \nu_o & \nu_1 & \mu_o & 0 \\ \eta_1 & \eta_2 & 0 & 0 & -\nu_1 & \nu_2 & 0 & \mu_2 \end{pmatrix} \tag{3.5}$$

where

$$\mu_o = m_o = \int_0^\infty S(\omega)\, d\omega$$

$$\mu_2 = \int_0^\infty (\omega-\bar{\omega})^2 S(\omega)\, d\omega$$

$$\bar{\omega} = \int_0^\infty \omega\, S(\omega)\, d\omega \Big/ \int_0^\infty S(\omega)\, d\omega$$

$$\nu_o = \int_0^\infty S(\omega) \cos(\omega-\bar{\omega})\tau \, d\omega$$

$$\nu_1 = \int_0^\infty S(\omega) (\omega-\bar{\omega}) \sin(\omega - \bar{\omega})\tau \, d\omega$$

$$\nu_2 = \int_0^\infty S(\omega) (\omega-\bar{\omega})^2 \cos(\omega-\bar{\omega})\tau \, d\omega \tag{3.6}$$

$$\eta_o = \int_0^\infty S(\omega) \sin(\omega-\bar{\omega})\tau \, d\omega$$

$$\eta_1 = \int_0^\infty S(\omega) (\omega-\bar{\omega}) \cos(\omega-\bar{\omega})\tau \, d\omega$$

$$\eta_2 = \int_0^\infty S(\omega)\ (\omega-\bar{\omega})^2 \sin(\omega-\bar{\omega})\ \tau\ d\omega$$

$$\tau = t_2 - t_1$$

$S(\omega)$ = wave specrum

The detailed derivation of the elements of the covariance matrix is given in Appendix A.

As can be seen in Eqs. (3.4) through (3.6), the joint probability density function f(**X**) can be evaluated from a given wave spectrum. In practice, however, the inverse operation of the covariance matrix involved in Eq. (3.4) is extremely difficult to perform. Even though the inverse can be achieved, the follow up transformation to the joint probability density function of the envelope and its velocity is not feasible.

One way to overcome this difficulty is to decompose a given spectrum into two parts, each part being symmetric about its mean frequency. One contains primarily the lower frequency energy components of the wave spectrum, while the other contains the higher frequencies of the spectrum as illustrated in Figure 3.1. Each part of the spectrum can be expressed in the form of

$$S_i(\omega) = a_i e^{-\pi \left(\frac{a_i}{m_{oi}}\right)^2 (\omega-\omega_{mi})^2} \tag{3.7}$$

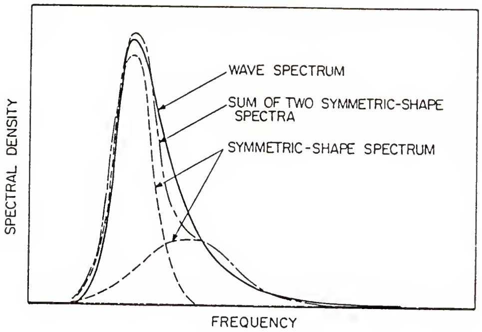


Figure 3.1 Decomposition of wave spectrum into two symmetric-shape spectra.

where $i = 1, 2$

ω_{mi} = modal frequency of $S_i(\omega)$

a_i = value of $S_i(\omega)$ at the modal frequency

m_{oi} = area under the spectrum $S_i(\omega)$.

The sum of the two areas m_{o1} and m_{o2} is equal to the area under the original spectrum $S(\omega)$ which, in turn is equal to the wave variance, μ_o. Since the variance of the velocity, μ_2, in the covariance matrix also plays a significant role in developing the probability density function of time duration $\tau_{\alpha+}$, the sum of the variances of the two decomposed spectra is maintained constant and is equal to the variance μ_2 of the original spectrum. Under these conditions, the parameters a_i, ω_{mi}, and m_{oi}, are determined numerically through a nonlinear least squares fitting technique such that the difference between the shape of the original spectrum and the sum of the two symmetric spectra, expressed by $|S(\omega) - \sum_{i=1}^{2} S_i(\omega)|^2$, is minimal. A detailed discussion on the decomposition of the spectrum is given in Appendix B.

Figure 3.2 shows an example of wave spectrum expressed by the sum of two symmetric spectra. The original spectrum represents the two-parameter wave spectral formulation for a significant wave height of 10.0 m with a modal frequency of 0.42 rps. Although there is some discrepancy in the magnitude of spectral densities between the original and the combined symmetric spectra at frequencies ranging from 0.60 and 0.85 rps, the values of moments which are significant for the analytical solution of wave groups are very close. That is, $\mu_o = 6.25\ m^2$ and $\mu_2 = 0.317\ m^2/sec^2$ of the combined spectra are chosen to be equal to those of the original spectrum. The computed moments m_1 and m_2 are 3.36

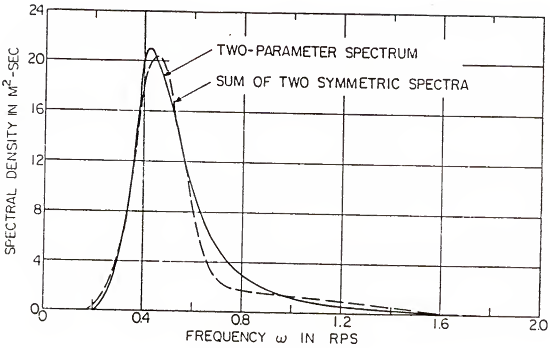


Figure 3.2 Comparison of two-parameter wave spectra (significant wave height 10.0 m., modal frequency 0.42 r.p.s.) and sum of two symmetric-shape spectra.

m^2/sec and 2.22 m^2/sec^2, respectively, for the combined spectra as compared with 3.40 m^2/sec and 2.19 m^2/sec^2, respectively, for the original spectrum.

The significance of representing the shape of a given wave spectrum by the sum of two symmetric spectra is that the covariance matrix given in Eq. (3.5) can be drastically simplified for each spectrum such that all covariances between x_c and x_s become zero (see Appendix C). That is, the covariance matrix of waves for each symmetric spectrum becomes

$$\Sigma_i = \begin{pmatrix} \mu_{oi} & 0 & \nu_{oi} & -\nu_{1i} & 0 & 0 & 0 & 0 \\ 0 & \mu_{2i} & \nu_{1i} & \nu_{2i} & 0 & 0 & 0 & 0 \\ \nu_{oi} & \nu_{1i} & \mu_{oi} & 0 & 0 & 0 & 0 & 0 \\ -\nu_{1i} & \nu_{2i} & 0 & \mu_{2i} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \mu_{oi} & 0 & \nu_{oi} & -\nu_{1i} \\ 0 & 0 & 0 & 0 & 0 & \mu_{2i} & \nu_{1i} & \nu_{2i} \\ 0 & 0 & 0 & 0 & \nu_{oi} & \nu_{1i} & \mu_{oi} & 0 \\ 0 & 0 & 0 & 0 & -\nu_{1i} & \nu_{2i} & 0 & \mu_{2i} \end{pmatrix} \qquad (3.8)$$

where i = 1 and 2, and elements in the above covariance matrix are evaluated for each decomposed spectrum. As can be seen in the covariance matrix, the sets of random variables $(x_{c1}, \dot{x}_{c1}, x_{c2}, \dot{x}_{c2})$ and $(x_{s1}, \dot{x}_{s1}, x_{s2}, \dot{x}_{s2})$ are uncorrelated and this makes further analytical development of the distribution function feasible.

We now consider two sets of random variables $(x_{c1}, \dot{x}_{c1}, x_{c2}, \dot{x}_{c2})$ and $(x_{s1}, \dot{x}_{s1}, x_{s2}, \dot{x}_{s2})$ both normally distributed with zero mean and assumed to be statistically independent. Hence, the sum of these two sets of random variables has a normal distribution with zero mean and a covariance matrix which is equal to the sum of the two covariance

matrices evaluated for the symmetric spectra. Hereafter, each element of the covariance matrix given in Eq. (3.5) represents the sum of the two elements in Eq. (3.8) computed for each symmetric spectrum. All η in the covariance matrix given in Eq. (3.5) are zero.

Since the inverse of the covariance matrix can be evaluated, the joint probability density function of $(x_{c1}, \dot{x}_{c1}, \text{--------} \dot{x}_{s2})$ can now be written as follows:

$$f(x_{c1}, \dot{x}_{c1}, x_{c2}, \dot{x}_{c2}, x_{s1}, \dot{x}_{s1}\ x_{s2}, \dot{x}_{s2}) = \frac{1}{(2\pi)^4} \cdot \frac{1}{|A|} e^{-\frac{K}{2}} \qquad (3.9)$$

where $$-\infty < \text{all } x, \dot{x} < \infty$$

$$A = \{(\mu_2+\nu_2)(\mu_o-\nu_o)-\nu_1^2\}\{(\mu_2-\nu_2)(\mu_o+\nu_o)-\nu_1^2\} \qquad (3.10)$$

$$K = \frac{1}{|A|} M_{11}(x_{c1}^2+x_{c2}^2+x_{s1}^2+x_{s2}^2) + M_{22}(\dot{x}_{c1}^2+\dot{x}_{c2}^2+\dot{x}_{s1}^2+\dot{x}_{s2}^2)$$

$$+ 2M_{12}(x_{c1}\dot{x}_{c1}-x_{c2}\dot{x}_{c2}+x_{s1}\dot{x}_{s1}-x_{s2}\dot{x}_{s2}) + 2M_{13}(x_{c1}x_{c2}+x_{s1}x_{s2})$$

$$+ 2M_{24}(\dot{x}_{c1}\dot{x}_{c2}+\dot{x}_{s1}\dot{x}_{s2}) + 2M_{14}(x_{c1}\dot{x}_{c2}-x_{c2}\,\dot{x}_{c1}+x_{s1}\dot{x}_{s2}-x_{s2}\,\dot{x}_{s1}) \qquad (3.11)$$

and $$M_{11} = \mu_o(\mu_2^2-\nu_2^2)-\nu_1^2\mu_2$$

$$M_{12} = \nu_1(\nu_o\mu_2-\nu_2\,\mu_o)$$

$$M_{13} = -\nu_o(\mu_2^2-\nu_2^2)+\nu_1^2\,\nu_2 \qquad (3.12)$$

$$M_{14} = \nu_1(\mu_o\mu_2-\nu_1^2-\nu_o\nu_2)$$

$$M_{22} = \mu_2(\mu_o^2 - \nu_o^2) - \nu_1^2 \mu_o$$

$$M_{24} = -\nu_2(\mu_o^2 - \nu_o^2) + \nu_o \nu_1^2$$

The detailed derivation of Eq. (3.9) is given in Appendix D.

Next, by applying the polar coordinates

$$\begin{aligned} x_{c1} &= R_1 \cos \Theta_1 \\ x_{s1} &= R_1 \sin \Theta_1 \\ x_{c2} &= R_2 \cos \Theta_2 \\ x_{s2} &= R_2 \sin \Theta_2, \end{aligned} \tag{3.13}$$

the random variables in Eq. (3.9) are transformed to a set of new random variables (R_1, $\dot{R}_1$, R_2, $\dot{R}_2$, Θ_1, $\dot{\Theta}_1$, Θ_2, $\dot{\Theta}_2$). Since R_1 and R_2 represent the envelope of a random process $x(t)$ at times t_1 and t_2, respectively, we set $R_1 = R_2 = \alpha$ for the problem at issue. Then, the joint probability density function of $\dot{R}_1$, $\dot{R}_2$, Θ_1, $\dot{\Theta}_1$, Θ_2, and $\dot{\Theta}_2$ for a specified α can be written by

$$f(\dot{R}_1, \dot{R}_2, \Theta_1, \dot{\Theta}_1, \Theta_2, \dot{\Theta}_2; \alpha) = \frac{\alpha^4}{(2\pi)^4} \cdot \frac{1}{|\mathbf{A}|} e^{-\frac{L}{2|\mathbf{A}|}} \tag{3.14}$$

$$0 \leqq \dot{R}_1 < \infty, \quad -\infty < \dot{R}_2 \leqq 0, \quad 0 \leqq \Theta_1, \Theta_2 \leqq 2\pi, \quad -\infty < \dot{\Theta}_1, \dot{\Theta}_2 < \infty$$

where

$$L = 2\alpha^2\{M_{11} + M_{13} \cos(\Theta_1 - \Theta_2)\} + 2\alpha \{M_{12} - M_{14} \cos(\Theta_1 - \Theta_2)\} (\dot{R}_1 - \dot{R}_2)$$

$$+ M_{22}(\dot{R}_1^2 + \dot{R}_2^2) + 2M_{24}\dot{R}_1\dot{R}_2 \cos(\Theta_1 - \Theta_2) + M_{22}\alpha^2(\dot{\Theta}_1^2 + \dot{\Theta}_2^2)$$

$$+ 2M_{24}\alpha^2 \dot{\theta}_1\dot{\theta}_2 \cos(\theta_1 - \theta_2) + 2M_{24}\alpha \; (\dot{R}_1\dot{\theta}_2 - \dot{R}_2 \dot{\theta}_1) \sin(\theta_1 - \theta_2)$$

$$+ 2M_{14}\alpha^2 \; (\dot{\theta}_1 + \dot{\theta}_2) \sin(\theta_1 - \theta_2) \tag{3.15}$$

It is noted that M_{11}, M_{22}, M_{13}, etc. in Eq. (3.15) are functions of ν_o, ν_1 and ν_2 as shown in Eq. (3.12), which in turn are functions of the time duration $\tau_{\alpha+}$ associated with the envelope exceeding the level α.

Next, Eq. (3.14) is integrated with respect to θ_1 and θ_2. The exponential part of Eq. (3.14) consists of many terms containing $\cos(\theta_1 - \theta_2)$ and $\sin(\theta_1 - \theta_2)$ as shown in Eq. (3.15); hence, the integration is carried out with respect to θ_1 and $(\theta_1 - \theta_2)$. The results yield

$$f(\dot{R}_1, \dot{R}_2, \dot{\theta}_1, \dot{\theta}_2; \alpha) = \frac{\alpha^4}{(2\pi)^3 \; |A|} \int_0^{2\pi} e^{-p} \, d\,\phi \tag{3.16}$$

where

$$P = \frac{1}{2|A|} \Big[2\,(M_{11} + M_{13}\cos\phi)\,\alpha^2 + 2(M_{12} - M_{14}\cos\phi)\,\alpha\,(\dot{R}_1 - \dot{R}_2)$$

$$+ M_{22}(\dot{R}_1^2 + \dot{R}_2^2) + 2M_{24}\,\dot{R}_1\dot{R}_2 \cos\phi \; + M_{22}\alpha^2(\dot{\theta}_1^2 + \dot{\theta}_2^2)$$

$$+ 2M_{24}\alpha^2 \,\dot{\theta}_1\dot{\theta}_2 \cos\phi + \; 2M_{24}\alpha \sin\phi \,(\dot{R}_1\dot{\theta}_2 - \dot{R}_2\dot{\theta}_1)$$

$$+ 2M_{14}\alpha^2 \,(\dot{\theta}_1 + \dot{\theta}_2) \sin\phi \Big] \tag{3.17}$$

After much lengthy mathematical manipulation, the details of which are given in Appendix E, the joint probability density function $f(\dot{R}_1,$

$\dot{R}_2$, $\dot{\Theta}_1$, $\dot{\Theta}_2$; α) is integrated with respect to $\dot{\Theta}_1$and $\dot{\Theta}_2$, and then, the joint probability density function $f(\dot{R}_1, \dot{R}_2; \alpha)$ can be written as

$$f(\dot{R}_1, \dot{R}_2; \alpha) = (\frac{\alpha}{2\pi})^2 \int_o^{2\pi} \frac{1}{M_{22}\sqrt{1-\rho^2}} \cdot \exp\left\{ - \left[\frac{\alpha^2(\mu_o - \nu_o \cos\phi)}{\mu_o^2 - \nu_o^2} + \frac{\mu_o^2 - \nu_o^2}{2M_{22}(1-\rho^2)} \{ (\dot{R}_1^2 + 2\rho \dot{R}_1\dot{R}_2 + \dot{R}_2^2) - 2k(1-\rho)(\dot{R}_1 - \dot{R}_2 - k)\} \right] \right\} d\phi \quad (3.18)$$

where

$$k = \alpha \frac{\nu_1(\mu_o \cos\phi - \nu_o)}{\mu_o^2 - \nu_o^2}$$

$$\rho = \frac{M_{24}}{M_{22}} \cos\phi = \frac{-\nu_2(\mu_o^2 - \nu_o^2) + \nu_o\nu_1^2}{\mu_2(\mu_o^2-\nu_o^2) - \nu_1^2\mu_o} \cos\phi \quad (3.19)$$

In order to carry out further integration necessary for the derivation of the probability density function of the time duration, $f(\tau_{\alpha+})$, the following transformation of the random variables is made:

$$\dot{R}_1 = \sqrt{\frac{M_{22}}{\mu_o^2 - \nu_o^2}}\, U = \sqrt{\mu_2 - \frac{\mu_o \nu_1^2}{\mu_o^2 - \nu_o^2}}\, U$$

$$\dot{R}_2 = -\sqrt{\frac{M_{22}}{\mu_o^2 - \nu_o^2}}\, V = -\sqrt{\mu_2 - \frac{\mu_o \nu_1^2}{\mu_o^2 - \nu_o^2}}\, V \quad (3.20)$$

Then, the joint probability density function of U and V for a specified α becomes

$$f(u,v;\alpha) = \left(\frac{\alpha}{2\pi}\right)^2 \frac{1}{\mu_o^2 - \nu_o^2} \int_0^{2\pi} \frac{1}{\sqrt{1-\rho^2}}$$

$$\times \exp\left\{ - \left[\frac{\alpha^2(\mu_o - \nu_o \cos\phi)}{\mu_o^2 - \nu_o^2} + \frac{1}{2(1-\rho^2)}(u^2 - 2\rho\, u\, v + v^2) \right.\right.$$

$$\left.\left. - \frac{1}{1+\rho}\left\{ (u+v)\left(k\sqrt{\frac{\mu_o^2-\nu_o^2}{M_{22}}}\right) - \left(k\sqrt{\frac{\mu_o^2-\nu_o^2}{M_{22}}}\right)^2 \right\}\right]\right\} d\phi \quad (3.21)$$

$$0 \leqq u < \infty$$

$$0 \leqq v < \infty$$

By applying the property associated with the bi-variate normal probability distribution (Cramér, 1946), the 2nd term of the exponential expression in Eq. (3.21) can approximately be written as follows (see Appendix F):

$$e^{-\frac{1}{2(1-\rho^2)}(u^2 - 2\rho\, u\, v + v^2)} \backsim \sqrt{1-\rho^2}\; e^{-\frac{u^2+v^2}{2}} \quad (3.22)$$

This approximation method was used by Tikhonov (1956) in the derivation of the probability distribution of time duration associated with excursion of Gaussian noise. The joint probability density function $f(u,v;\alpha)$ then approximately becomes

$$f(u, v; \alpha) = \left(\frac{\alpha^2}{2\pi}\right) \frac{1}{\mu_o^2 - \nu_o^2} \int_o^{2\pi} \exp\left\{ - \left[\frac{\alpha^2(\mu_o - \nu_o \cos\phi)}{\mu_o^2 - \nu_o^2} + \frac{u^2+v^2}{2} \right.\right.$$

$$\left.\left. - \frac{1}{1+\rho}\left\{ (u+v)\left(k\sqrt{\frac{\mu_o^2-\nu_o^2}{M_{22}}}\right) - \left(k\sqrt{\frac{\mu_o^2-\nu_o^2}{M_{22}}}\right)^2 \right\}\right]\right\} d\phi \quad (3.23)$$

By transformation of the random variables $\dot{R}_1$ and $\dot{R}_2$ to U and V, respectively, the numerator of Eq. (3.1) can be expressed in terms of U and V as follows:

$$\bar{N}_{\alpha+}(\tau_{\alpha+}) = - \int_{-\infty}^{0} \int_{0}^{\infty} \dot{R}_1 \dot{R}_2 \; f(\dot{R}_1, \dot{R}_2; \alpha) \; d\dot{R}_1 d\dot{R}_2$$

$$= \frac{M_{22}}{\mu_o^2 - \nu_o^2} \int_0^{\infty} \int_0^{\infty} u \; v \; f(u, v; \alpha) \; du \; dv \tag{3.24}$$

Then, from Eqs. (3.23) and (3.24), we have

$$\bar{N}_{\alpha+}(\tau_{\alpha+}) = \left(\frac{\alpha}{2\pi}\right)^2 \frac{M_{22}}{(\mu_o^2 - \nu_o^2)^2}$$

$$\times \int_0^{2\pi} \left(\int_0^{\infty} \int_0^{\infty} u \; v \; \exp\left\{ - \left[\frac{u^2 + v^2}{2} - (u + v) \frac{k}{1+\rho} \sqrt{\frac{\mu_o^2 - \nu_o^2}{M_{22}}} \right] \right\} du \; dv \right)$$

$$\times \exp\left\{ - \left[\frac{\alpha^2(\mu_o - \nu_o \cos \phi)}{\mu_o^2 - \nu_o^2} + \frac{1}{1+\rho} \left(k \sqrt{\frac{\mu_o^2 - \nu_o^2}{M_{22}}} \right)^2 \right] \right\} d\phi \tag{3.25}$$

Since the terms associated with u and v in Eq. (3.25) are symmetric, we can simply write

$$\int_0^{\infty} \int_0^{\infty} u \; v \; \exp\left\{ - \left[\frac{u^2 + v^2}{2} - (u + v) \frac{k}{1+\rho} \sqrt{\frac{\mu_o^2 - \nu_o^2}{M_{22}}} \right] \right\} du \; dv \tag{3.26}$$

$$=\left(\int_0^{\infty} u \exp\left\{-\left[\frac{u^2}{2} - u\frac{k}{1+\rho}\sqrt{\frac{\mu_o^2-\nu_o^2}{M_{22}}}\right]\right\} du\right)^2$$

By letting

$$\frac{k}{1+\rho}\sqrt{\frac{\mu_o^2-\nu_o^2}{M_{22}}} = 2\gamma \tag{3.27}$$

the integration given in Eq. (3.26) can be obtained as

$$\int_o^{\infty} u\, e^{-(\frac{u^2}{2} - 2\gamma u)} du = 1 + \sqrt{2\pi}\,\gamma\, e^{2\gamma^2}\left\{1 + \Phi(\sqrt{2}\,\gamma)\right\}$$

where
$$\Phi(z) = \frac{2}{\sqrt{\pi}}\int_0^z e^{-t^2} dt \tag{3.28}$$

Hence, Eq. (3.25) can be evaluated by

$$\bar{N}_{\alpha+}(\tau_{\alpha+}) = \left(\frac{\alpha}{2\pi}\right)^2 \frac{M_{22}}{(\mu_o^2-\nu_o^2)^2}\int_o^{2\pi} \exp\left\{-\left[\frac{\alpha^2(\mu_o-\nu_o\cos\phi)}{\mu_o^2-\nu_o^2} + 4(1+\rho)\gamma^2\right]\right\}$$

$$\times\left(1 + \sqrt{2\pi}\,\gamma\, e^{2\gamma^2}\left\{1 + \Phi(\sqrt{2}\,\gamma)\right\}\right)^2 d\phi$$

$$= \left(\frac{\alpha}{2\pi}\right)^2 \frac{M_{22}}{(\mu_o^2-\nu_o^2)^2}\int_o^{2\pi} \exp\left\{-\left[\frac{\alpha^2(\mu_o-\nu_o\cos\phi)}{\mu_o^2-\nu_o^2} + 4\rho\gamma^2\right]\right\}$$

$$\times\left(e^{-2\gamma^2} + \sqrt{2\pi}\,\gamma\left\{1 + \Phi(\sqrt{2}\,\gamma)\right\}\right)^2 d\phi$$

$$= \left(\frac{\alpha}{2\pi}\right)^2 \frac{M_{22}}{(\mu_o^2-\nu_o)^2}\int_o^{2\pi} \exp\left\{-\left[\frac{\alpha^2(\mu_o-\nu_o\cos\phi)}{\mu_o^2-\nu_o^2} + 4\rho\gamma^2\right]\right\}$$

$$\times \left[\begin{array}{l} \left(e^{-2\gamma^2} - \sqrt{2\pi}\ \gamma\ \{1 - \Phi(\sqrt{2}\,\gamma)\}\right)^2 \\ \\ + 4\sqrt{2}\,\pi\,\gamma \left\{ \dfrac{1}{\sqrt{\pi}}\ e^{-2\gamma^2} + \sqrt{2}\ \gamma\ \Phi(\sqrt{2}\,\gamma) \right\} \end{array} \right] d\,\phi \tag{3.29}$$

The first term of the above equation represents the expected number (per unit time) of the envelope downward crossing with velocity $\dot{R}_1$ followed by the upward crossing with velocity $\dot{R}_2$, namely $\bar{N}_{\alpha-}(\tau_{\alpha-})$ (see Appendix G). Since $\bar{N}_{\alpha-}(\tau_{\alpha-})$ is smaller than $\bar{N}_{a+}(\tau_{a+})$ for a large α, the first term of Eq. (3.29) can be neglected. Thus, we can write

$$\bar{N}_{\alpha+}(\tau_{\alpha+}) = \frac{\sqrt{2}\,\alpha^2}{\pi}\ \frac{M_{22}}{(\mu_o^2 - \nu_o^2)^2} \int_o^{2\pi} \exp\left\{ - \left[\frac{\alpha^2(\mu_o - \nu_o\cos\phi)}{\mu_o^2 - \nu_o^2} + 4\rho\gamma^2 \right] \right\}$$
$$\times\ \gamma \left\{ \frac{1}{\sqrt{\pi}}\ e^{-2\gamma^2} + \sqrt{2}\ \gamma\ \Phi(\sqrt{2}\,\gamma) \right\} d\,\phi \tag{3.30}$$

On the other hand, the denominator of Eq. (3.1) becomes

$$\bar{N}_{\alpha+} = \int_o^{\infty} \dot{R}_1\ f(\dot{R}_1;\ \alpha)\ d\dot{R}_1 = \sqrt{\frac{\mu_2}{2\pi}}\ \frac{\alpha}{\mu_o}\ e^{-\frac{\alpha^2}{2\mu_o}} \tag{3.31}$$

As stated in regard to Eq. (3.1), the equation is equivalent to the probability density function of the time duration, $\tau_{\alpha+}$, associated with the envelope exceeding a specified level α. Thus, from Eqs. (3.30) and (3.31), the probability density function of $\tau_{\alpha+}$ can be obtained as

$$f(\tau_{\alpha+}) = \frac{2\,\alpha\,\mu_o}{\sqrt{\pi}\,\mu_2} \quad \frac{M_{22}}{(\mu_o^2 - \nu_o^2)^2}\int_0^{2\pi} \exp\left\{-\left[\frac{\alpha^2(\mu_o - \nu_o\cos\phi)}{\mu_o^2 - \nu_o^2} - \frac{\alpha^2}{2\mu_o} + 4\,\rho\,\gamma^2\right]\right\}$$

$$\times\,\gamma\left\{\frac{1}{\sqrt{\pi}}\,e^{-2\gamma^2} + \sqrt{2}\,\gamma\,\Phi(\sqrt{2}\,\gamma)\right\} d\,\phi \tag{3.32}$$

It is assumed that ϕ is relatively small for large α , and thereby we have the approximation $\cos\phi \sim 1 - \phi^2/2$. Then, Eq. (3.32) may be written as

$$f(\tau_{\alpha+}) = C\,\frac{2\alpha\mu_o}{\sqrt{\pi}\,\mu_2} \quad \frac{M_{22}}{(\mu_o^2 - \nu_o^2)^2}\int_0^{2\pi} \exp\left\{-\left[\frac{\alpha^2\{\mu_o - \nu_o(1 - \phi^2/2)\}}{\mu_o^2 - \nu_o^2} - \frac{\alpha^2}{2\mu_o} + 4\,\rho\,\gamma'^2\right]\right\}$$

$$\times\,\gamma'\left\{\frac{1}{\sqrt{\pi}}\,e^{-2\gamma'^2} + \sqrt{2}\,\gamma'\,\Phi(\sqrt{2}\,\gamma')\right\} d\,\phi \tag{3.33}$$

where

$$\gamma' = \frac{\alpha}{2(1 + \rho)\sqrt{M_{22}}} \quad \frac{\nu_1\{\mu_o(1 - \phi^2/2) - \nu_o\}}{\sqrt{\mu_o^2 - \nu_o^2}} \tag{3.34}$$

C = normalization factor to make the area under the density function unity

$= 1/\int_0^\infty f(\tau_{\alpha+})\,d\tau_{\alpha+}$

3.2 Probability Distribution of Time Duration Associated With Wave Group

In this section the probability density function of the time duration associated with wave groups will be derived. The probability density function derived in the previous section is that applicable to the time duration that the envelope of a narrow-band random process exceeds a specified level. If the time duration is not sufficiently long, then there may be no wave crest or only one wave crest occurring during the time the envelope exceeds the specified level, and this cannot be considered as a wave group. Therefore, the probability density function of the time interval associated with the envelope exceeding a specified level should be modified so that it is only concerned with the time interval during which two or more wave crests are present. For this, let us first examine the relationship between the time duration $\tau_{\alpha+}$ and the number of wave crests involved in $\tau_{\alpha+}$.

From the assumption of a narrow-band random process, the time interval between two successive wave crests associated with wave groups is considered to be equal to the average zero-crossing period, $\bar{T}_o$, which can be evaluated by the following formula:

$$\bar{T}_o = 2\pi \sqrt{m_o / m_2} \tag{3.35}$$

where $\qquad m_j$ = j-th moment of wave spectrum.

Then, we can evaluate the number of waves during the time interval $\tau_{\alpha+}$ based on the average zero-crossing period as follows:

(i) $t_{\alpha+} \leqq \bar{T}_o$: If the time duration of the envelope, $\tau_{\alpha+}$, is less than $\bar{T}_o$, there is either no wave crest or only one wave crest in $\tau_{\alpha+}$. Hence, this situation cannot be considered as a wave group although the envelope exceeds the specified level α. The probability of occurrence of this situation can be evaluated from Eq. (3.34) as

$$p_o = \int_o^{\bar{T}_o} f(\tau_{\alpha+}) \; d\tau_{\alpha+} \tag{3.36}$$

(ii) $\bar{T}_o \leqq \tau_{\alpha+} \leqq 2\,\bar{T}_o$: If the time duration $\tau_{\alpha+}$ is between $\bar{T}_o$ and $2\,\bar{T}_o$, then there is either one wave crest or two wave crests during $\tau_{\alpha+}$. The probabilities of one and two wave crests during $\tau_{\alpha+}$ can be evaluated by

$$\Pr\left\{\text{One wave crest} \mid \tau_{\alpha+}\right\} = 1 - 2\left(1 - \frac{\bar{T}_o}{\tau_{\alpha+}}\right) \tag{3.37}$$

$$\Pr\left\{\text{Two wave crests} \mid \tau_{\alpha+}\right\} = 2\left(1 - \frac{\bar{T}_o}{\tau_{\alpha+}}\right) \tag{3.38}$$

Since one wave crest in $\tau_{\alpha+}$ cannot be considered as a wave group, this situation should be eliminated from the probability density function for the time duration associated with wave groups. The probability of occurrence of one wave crest in $\bar{T}_o \leqq \tau_{\alpha+} \leqq 2\,\bar{T}_o$ can be evaluated by

$$p_1 = \int_{\bar{T}_o}^{2\bar{T}_o} \left\{1 - 2\left(1 - \frac{\bar{T}_o}{\tau_{\alpha+}}\right)\right\} f(\tau_{\alpha+}) \; d\tau_{\alpha+} \tag{3.39}$$

(iii) $2\bar{T}_o \leqq \tau_{\alpha+} \leqq 3\bar{T}_o$: Under this condition, there are either two or three wave crests in time duration $\tau_{\alpha+}$; hence, this situation can be considered a wave group.

Thus, in summary, an exceedance of the envelope above the specified level does not necessarily mean wave groups, since there may be no wave crest or only one crest if the exceeding time duration is not sufficiently long. The probability of occurrence of this situation is given by $(p_o + p_1)$ as shown in the above. Therefore, the probability of wave group when envelope exceeds the specified level α is given by the following formula:

$$\Pr\left\{\begin{matrix}\text{Wave groups when envelope}\\ \text{exceeds a level } \alpha\end{matrix}\right\} = 1 - (p_o + p_1)$$

$$= 1 - \left[\int_0^{\bar{T}_o} f(\tau_{\alpha+})\, d\tau_{\alpha+} + \int_{\bar{T}_o}^{2\bar{T}_o} \left\{1 - 2\left(1 - \frac{\bar{T}_o}{\tau_{\alpha+}}\right)\right\} f(\tau_{\alpha+})\, d\tau_{\alpha+}\right] \quad (3.40)$$

The probability density function of time duration associated with wave groups can be obtained from the probability density function $f(\tau_{\alpha+})$ by eliminating the portion in which either no wave crest or one crest occurs in $\tau_{\alpha+}$. That is, the entire probability density function for $\tau_{\alpha+} \leqq \bar{T}_o$ is discarded, and the magnitude of the density function for $\bar{T}_o \leqq \tau_{\alpha+} \leqq 2\bar{T}_o$ is modified by multiplying by the probability given in Eq. (3.38). This is shown in the pictorial sketch given in Figure 3.3. Then, the modified probability density function is normalized so that the area under the density function becomes unity. The probability density function thus derived, denoted by $f(\tau_G)$, which

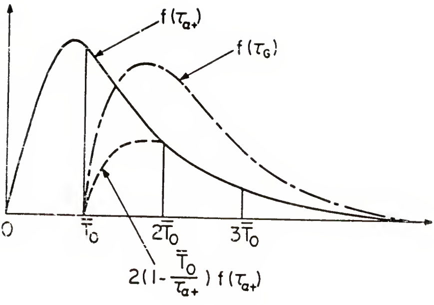


Figure 3.3 Sketch showing the relationship between probability density function associated with wave envelope, $f(\tau_{\alpha+})$, and that associated with wave groups, $f(\tau_G)$.

is applicable to the time duration associated with wave groups is summarized as follows:

$$f(\tau_G) = \begin{cases} 0 & \text{for } 0 \leqq \tau_G \leqq \bar{T}_o \\ \dfrac{1}{1-(p_o+p_1)} \; 2\left(1-\dfrac{\bar{T}_o}{\tau_G}\right)\left| f(\tau_{\alpha+})\right|_{\tau_{\alpha+}=\tau_G} & \text{for } \bar{T}_o \leqq \tau_G \leqq 2\bar{T}_o \\ \dfrac{1}{1-(p_o+p_1)} \left| f(\tau_{\alpha+})\right|_{\tau_{\alpha+}=\tau_G} & \text{for } 2\,\bar{T}_o \leqq \tau_G \end{cases} \tag{3.41}$$

The expected (average) time duration associated with wave groups, denoted by $\bar{\tau}_G$, can thereby be evaluated as

$$\bar{\tau}_G = \int_o^{\infty} \tau_G \; f(\tau_G)\, d\tau_G$$

$$= \frac{1}{1-(p_o+p_1)}\left\{\int_{\bar{T}_o}^{2\bar{T}_o} 2\,\tau_G\left(1-\frac{\bar{T}_o}{\tau_G}\right)\left|f(\tau_{\alpha+})\right|_{\tau_{\alpha+}=\tau_G} d\tau_G \right.$$

$$\left. + \int_{2\bar{T}_o}^{\infty} \tau_G \left|f(\tau_{\alpha+})\right|_{\tau_{\alpha+}=\tau_G} d\tau_G \right\} \tag{3.42}$$

3.3 Number of Wave Crests in a Group

Once the probability function of the time duration associated with wave groups is derived, statistical information on the number of waves involved in a group can be obtained therefrom. For example, the probabilty of occurrence of a specified number of waves in a group can be evaluated from the probability density function developed in the previous section.

Let us first evaluate the average (mean) number of wave crests in a wave group for a given wave spectrum. Assuming that the time interval between two successive wave crests is equal to the average zero-crossing period, the average number of wave crests can be obtained from Eqs. (3.35) and (3.42), as

$$\text{Average number of wave crests} = \bar{\tau}_G / \bar{T}_o \qquad (3.43)$$

Next, let us evaluate the probability of two wave crests, three wave crests, etc., in a group. In general, for the time duration $(m-1)\,\bar{T}_o \leqq \tau_G \leqq m\bar{T}_o$, where m is an integer 2,3,----, there are either (m-1) waves or m-waves in τ_G. The probability of m-waves in a specified τ_G is given by

$$\Pr\left\{ \text{m-waves} \mid \tau_G \right\} = m\left(1 - \frac{\bar{T}_o}{\tau_G}\right) \qquad (3.44)$$

where $(m-1)\,\bar{T}_o \leqq \tau_G \leqq m\bar{T}_o$

$\bar{T}_o$ = average zero-crossing wave period

The probability of (m-1) waves for a given τ_G becomes

$$\Pr\left\{(m-1)\text{-waves} \,\middle|\, \tau_G\right\} = 1 - m\left(1 - \frac{\bar{T}_o}{\tau_G}\right) \tag{3.45}$$

From the conditional probability given above, the probability of occurrence of m-waves in τ_G, where $(m-1)\bar{T}_o \leqq \tau_G \leqq m\bar{T}_o$, can be evaluated by

$$\Pr\left\{m\text{-waves}\right\} = \int_{(m-1)\bar{T}_o}^{m\bar{T}_o} m\left(1 - \frac{\bar{T}_o}{\tau_G}\right) f(\tau_G)\, d\tau_G \tag{3.46}$$

Similarly, we have

$$\Pr\left\{(m-1)\text{ waves}\right\} = \int_{(m-1)\bar{T}_o}^{m\bar{T}_o} \left\{1 - m\left(1 - \frac{\bar{T}_o}{\tau_G}\right)\right\} f(\tau_G)\, d\tau_G \tag{3.47}$$

By applying the formulae derived in Eqs. (3.46) and (3.47), we can evaluate from the wave spectrum the probability of occurrence of a specified number of waves in a group as follows:

$$\Pr\left\{\begin{array}{l}\text{Two waves}\\ \text{in a group}\end{array}\right\} = \int_{\bar{T}_o}^{2\bar{T}_o} 2\left(1 - \frac{\bar{T}_o}{\tau_G}\right) f(\tau_G)\, d\tau_G + \int_{2\bar{T}_o}^{3\bar{T}_o} \left\{1-3\left(1 - \frac{\bar{T}_o}{\tau_G}\right)\right\} f(\tau_G)\,d\tau_G$$

$$\Pr\left\{\begin{array}{c}\text{Three waves}\\ \text{in a group}\end{array}\right\} = \int_{2\bar{T}_o}^{3\bar{T}_o} 3\left(1 - \frac{\bar{T}_o}{\tau_G}\right) f(\tau_G)\,d\tau_G + \int_{3\bar{T}_o}^{4\bar{T}_o} \left\{1-4\left(1 - \frac{\bar{T}_o}{\tau_G}\right)\right\} f(\tau_G)\,d\tau_G$$

$$Pr\left\{\begin{matrix}\text{Four waves}\\ \text{in a group}\end{matrix}\right\} = \int_{3\bar{T}_o}^{4\bar{T}_o} 4\left(1 - \frac{\bar{T}_o}{\tau_G}\right) f(\tau_G)d\tau_G + \int_{4\bar{T}_o}^{5\bar{T}_o} \left\{1-5\left(1 - \frac{\bar{T}_o}{\tau_G}\right)\right\} f(\tau_G)d\tau_G$$

etc. (3.48)

3.4 Example of Application

As an example of the application of the prediction method for evaluating the time duration and number of waves associated with wave groups, numerical computations were carried out using the two-parameter wave spectrum shown in Figure 3.2 having a significant wave height of 10.0 m and a modal frequency of 0.42 rps.

The probability density function of the time duration of the envelope exceeding levels of 5.0, 6.0, 7.0, and 8.0 m above the mean water level was computed by Eq. (3.33), and the results are shown in Figure 3.4. The average values of the time duration $\bar{\tau}_{\alpha+}$ were then evaluated from Eq. (3.33) by

$$\bar{\tau}_{\alpha+} = \int_0^{\infty} \tau_{\alpha+} \; f(\tau_{\alpha+}) \; d\tau_{\alpha+} \tag{3.49}$$

and the results compared with those computed to date by the following formula:

$$\bar{\tau}_{\alpha+} = \frac{\int_{\alpha}^{\infty} f(R) \; dR}{\int_0^{\infty} \dot{R} \; f(\alpha, \dot{R}) \; d\dot{R}} = \sqrt{\frac{2\pi}{\mu_2}} \; \frac{\mu_o}{\alpha} \tag{3.50}$$

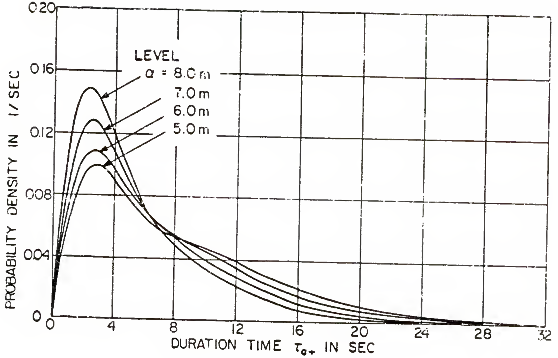


Figure 3.4 Probability density function of time duration associated with wave envelope at various levels.

where $f(\alpha,\dot{R})$ is the joint probability density function of amplitude and velocity of the envelope, and μ_o and μ_2 are as defined in Eq. (3.6).

As can be seen in Table 3.1, the average time duration for a given level evaluated by the probability density function derived in the present study (Eq. 3.49) is substantially greater than that computed by the formula given in Eq. (3.50).

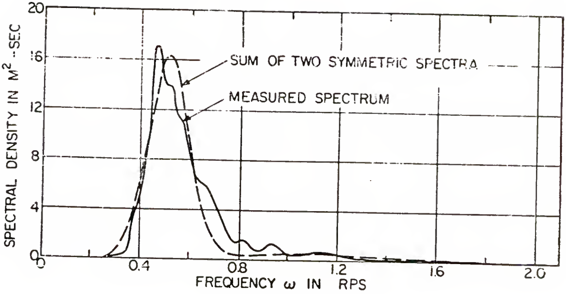
Additional computations were carried out using wave data measured at sea. The data were taken in North Sea off Norway by the Norwegian Marine Technology Research Institute. The water depth at the measured site is 230 meters. Two sets of data (significant wave heights of 8.13 m and 7.44 m) were analyzed. Figure 3.5 shows the measured spectrum as well as the sum of the two symmetric-shaped spectra used in the computations for each record.

Since the observation time for each record for which the seas were considered to be steady-state is not sufficiently long, the sample size of the envelope exceeding a specified level is rather small. Therefore, the average values obtained from the measured data may not be accurate. Nevertheless, a consistent trend can be seen in Table 3.2 where the measured and computed average time durations above various levels are tabulated. That is, although the average values obtained from the records are greater than those computed by Eq. (3.49), the two are very close for high crossing levels. On the other hand, the computed values by Eq. (3.50) are smaller than the values obtained from the records by a considerable amount.

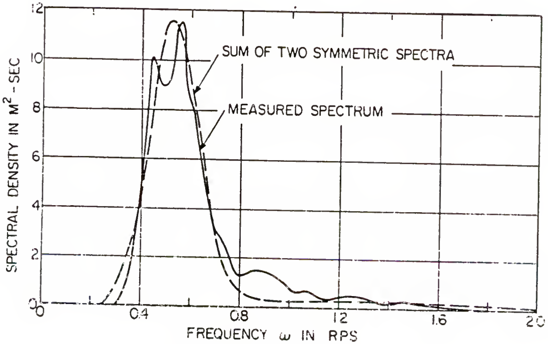
As stated in connection with Eq. (3.40), an exceedance of the envelope above a specified level does not necessarily mean a wave group. The probability of occurrence of wave when the envelope exceeds

Table 3.1 Comparison of average time duration associated with the envelope exceeding a specified level.

Level	Average time duration computed by	
	Eq. (3.49)	Eq. (3.50)
5.0 m.	8.2 sec.	5.6 sec.
6.0	7.3	4.6
7.0	6.4	4.0
8.0	5.5	3.5



(a) Wave spectrum A, significant wave height 8.13 m.



(b) Wave spectrum B, significant wave height 7.44 m.

Figure 3.5 Measured wave spectra used in analysis.

Table 3.2 Comparison between observed and computed average time durations associated with the envelope exceeding a specified level.

	Level	Observed		Computed average time duration	
		Sample size	Average time duration	Eq. (3.49)	Eq. (3.50)
Spectrum A $H_s = 8.13$ m.	4.0 m.	9	15.9 sec.	9.1 sec.	3.9 sec.
	5.0	6	10.8	7.9	3.1
	6.0	2	7.0	6.7	2.6
Spectrum B $H_s = 7.44$ m.	4.0 m.	7	10.7 sec.	7.4 sec.	3.7 sec.
	5.0	4	7.1	6.3	3.0

group. The probability of occurrence of wave when the envelope exceeds the specified level was calculated by Eq. (3.40) using the two-parameter wave spectrum and the results are tabulated in Table 3.3.

As can be seen in the table, the majority of the level crossings cannot be considered as indicative of the occurrence of wave groups since there is either no wave crest or only one crest during the crossing time interval.

Figure 3.6 shows the probability density function of the time duration associated with wave groups exceeding levels of 5.0, 6.0, 7.0, and 8.0 m above the mean water level computed by Eq. (3.41) using the two-parameter wave spectrum. It can be seen from a comparison of Figures 3.4 and 3.6 that there is a substantial difference in these probability density functions.

Table 3.4 shows the comparison between observed and computed average time durations associated with wave group for the measurerd wave spectrum A shown in Figure 3.5(a). Agreement between them appears to be satisfactory.

Table 3.3 Probability of occurrence of wave group when envelope exceeds a specified level.

Level	Probability of		
	Wave group	No wave group	
		No wave crest	One wave crest
5.0 m.	0.16	0.71	0.13
6.0	0.12	0.76	0.12
7.0	0.08	0.83	0.09
8.0	0.05	0.88	0.07

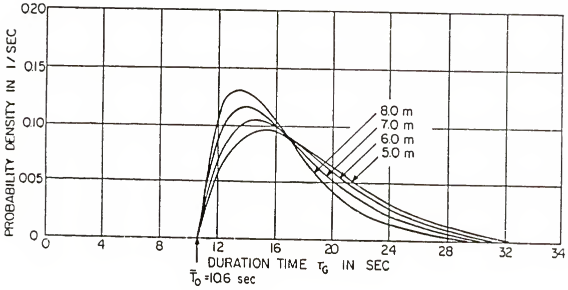


Figure 3.6 Probability density function of time duration associated with wave groups at various levels.

Table 3.4 Comparison between observed and computed average time durations associated with wave group (Spectrum A, H_s = 8.13 m.)

Level	Observed		Computed average time duration
	Sample size	Average time duration	
4.0 m.	6	19.0 sec.	17.9 sec.
5.0	1	13.7	17.0

CHAPTER IV
FREQUENCY OF OCCURRENCE OF WAVE GROUPS

4.1 Probability Distribution of Time Interval Between Successive Crossings of Specified Level by Envelope

We first consider the level crossing of the envelope of a narrow band random process and derive the probability distribution of the time interval between successive positive crossings of the envelope, R(t), at a specified level α which is denoted by τ_α as shown in Figure 1.3.

The distribution function of τ_α can be evaluated by applying the concept presented in Chapter III of the present study. That is, the probability distribution function of time duration of the envelope exceeding a specified level α, denoted by $\tau_{\alpha+}$, was developed in Chapter III. In a similar fashion, the probability distribution function of time duration of the envelope below a specified level, denoted by $\tau_{\alpha-}$, may be developed. Since the time interval between successive positive crossings of the envelope, τ_α, is the sum of $\tau_{\alpha+}$ and $\tau_{\alpha-}$, the probability distribution function of τ_α can be derived by the convolution integral of two probability density functions $f(\tau_{\alpha+})$ and $f(\tau_{\alpha-})$. This approach, however, has a drawback in the accuracy of the probability density function $f(\tau_{\alpha-})$. To elaborate, the following explanation is given:

The method for deriving the probability density function developed in Chapter III is valid for a relatively small time duration. Since a fairly high level of envelope crossings is considered for wave groups, the time duration $\tau_{\alpha+}$ is small, but the time duration of the envelope crossing below the level, $\tau_{\alpha-}$, is essentially large. Hence, there is no

assurance as to the accuracy of the probability density function $f(\tau_{\alpha-})$ if it is derived following the approach developed in Chapter III.

One way to overcome the difficulty involved in the derivation of the probability density function of the time interval between successive positive crossings of the envelope is to assume that the probability density function is approximately equal to that of the time interval between successive maxima of the envelope above a specified level α, denoted by τ_m in Figure 1.3. This assumption is permissible because the crossings take place at a fairly large distance above the zero-line for wave groups, and this results in the time interval between positive crossings being nearly equal to that between successive maxima of the envelope.

Following Rice's method developed for evaluating the average number of maxima per unit time (Rice, 1945), we assume waves to be a narrow-band random process whose profile x(t) can be written as

$$x(t) = \sum_{n=1}^{\infty} c_n \cos(\omega_n t + \varepsilon_n) = x_c(t) \cdot \cos \bar{\omega} t - x_s(t) \cdot \sin \bar{\omega} t \tag{4.1}$$

where

$$x_c(t) = \sum_{n=1}^{\infty} c_n \cos\{(\omega_n - \bar{\omega})t - \varepsilon_n\}$$

$$x_s(t) = \sum_{n=1}^{\infty} c_n \sin\{(\omega_n - \bar{\omega})t - \varepsilon_n\} \tag{4.2}$$

$$c_n = \sqrt{2\, S(\omega_n) \cdot \Delta\omega_n}$$

$\bar{\omega}$ = mean frequency of wave spectrum

$$= \int_0^{\infty} \omega\, S(\omega)\, d\omega \Big/ \int_0^{\infty} S(\omega)\, d\omega$$

Since we are interested in the maxima of the envelope shown in Figure 1.3, we consider a random vector **X** which is composed of a set of six random variables $x_c, x'_c, x''_c, x_s, x'_s, x''_s$ representing the cosine and sine components of displacement, velocity, and acceleration.

Each element of **X** obeys the normal probability distribution and we can write the probability density function of **X** in the following form:

$$f(\mathbf{X}) = \frac{1}{(2\pi)^3} \frac{1}{|\Sigma|^{1/2}} e^{-\frac{1}{2}\mathbf{X}'\Sigma^{-1}\mathbf{X}} \tag{4.3}$$

where $\mathbf{X}' = (x_c, x'_c, x''_c, x_s, x'_s, x''_s)$

Σ is the covariance matrix given by

$$\Sigma = \begin{pmatrix} \mu_o & 0 & -\mu_2 & 0 & 0 & 0 \\ 0 & \mu_2 & 0 & 0 & 0 & \mu_3 \\ -\mu_2 & 0 & \mu_4 & 0 & -\mu_3 & 0 \\ 0 & 0 & 0 & \mu_o & 0 & -\mu_2 \\ 0 & 0 & -\mu_3 & 0 & \mu_2 & 0 \\ 0 & \mu_3 & 0 & -\mu_2 & 0 & \mu_4 \end{pmatrix} \tag{4.4}$$

where $$\mu_r = \int_0^\infty (\omega - \bar{\omega})^r \cdot S(\omega)\, d\omega \tag{4.5}$$

$S(\omega)$ = wave spectrum

The derivation of elements of the covariance matrix associated with displacement and velocity was given in Appendix A. The derivation of the elements associated with acceleration is given in Appendix H.

As can be seen in Eq. (4.4), it is difficult to invert the covariance matrix Σ as needed in Eq. (4.3). However, by interchanging the

positions of $\dot{x}_c$ and $\dot{x}_s$, the matrix given in Eq. (4.4) can be written as

$$\Sigma = \begin{pmatrix} \mu_o & 0 & -\mu_2 & 0 & 0 & 0 \\ 0 & \mu_2 & -\mu_3 & 0 & 0 & 0 \\ -\mu_2 & -\mu_3 & \mu_4 & 0 & 0 & 0 \\ 0 & 0 & 0 & \mu_o & 0 & -\mu_2 \\ 0 & 0 & 0 & 0 & \mu_2 & \mu_3 \\ 0 & 0 & 0 & -\mu_2 & \mu_3 & \mu_4 \end{pmatrix} \tag{4.6}$$

and thereby the inversion of the matrix is feasible.

The joint probability density function of $(x_c, \dot{x}_s, \ddot{x}_c, x_s, \dot{x}_c, \ddot{x}_s)$ can now be written as follows:

$$f(x_c, \dot{x}_s, \ddot{x}_c, x_s, \dot{x}_c, \ddot{x}_s) = \frac{1}{(2\pi)^3} \frac{1}{|\mathbf{A}|} e^{-\frac{K}{2}} \tag{4.7}$$

$$-\infty < \text{all } x, \dot{x}, \ddot{x} < \infty$$

where $$|\mathbf{A}| = \mu_o(\mu_2 \mu_4 - \mu_3^2) - \mu_2^3$$

$$K = \frac{1}{|\mathbf{A}|} \{M_{11}(x_c^2 + x_s^2) + M_{22}(\dot{x}_c^2 + \dot{x}_s^2) + M_{33}(\ddot{x}_c^2 + \ddot{x}_s^2)$$

$$+ 2M_{12}(x_c \dot{x}_s - \dot{x}_c x_s) + 2M_{13}(x_c \ddot{x}_c + x_s \ddot{x}_s) + 2M_{23}(\ddot{x}_c \dot{x}_s - \dot{x}_c \ddot{x}_s)\}$$

(4.8)

and

$$
\begin{aligned}
M_{11} &= \mu_2\, \mu_4 - \mu_3^2 \\
M_{22} &= \mu_o\, \mu_4 - \mu_2^2 \\
M_{33} &= \mu_o\, \mu_2 \\
M_{12} &= \mu_2\, \mu_3 \\
M_{13} &= \mu_2^2 \\
M_{23} &= \mu_o\, \mu_3
\end{aligned}
\tag{4.9}
$$

The detailed derivation of Eq. (4.7) is given in Appendix I.

Next, by applying the following polar coordinates, the joint probability density function given in Eq. (4.7) will be transformed to that of R, $\dot{R}$, $\ddot{R}$, θ, $\dot{\theta}$, $\ddot{\theta}$. That is, by letting

$$
\begin{aligned}
x_c &= R\cos\theta \\
x_s &= R\sin\theta \\
\dot{x}_c &= \dot{R}\cos\theta - R\,\dot{\theta}\sin\theta \\
\dot{x}_s &= \dot{R}\sin\theta + R\,\dot{\theta}\cos\theta \\
\ddot{x}_c &= \ddot{R}\cos\theta - 2\,\dot{R}\,\dot{\theta}\sin\theta - R\,\dot{\theta}^2\cos\theta - R\,\ddot{\theta}\sin\theta \\
\ddot{x}_s &= \ddot{R}\sin\theta + 2\,\dot{R}\,\dot{\theta}\cos\theta - R\,\dot{\theta}^2\sin\theta + R\,\ddot{\theta}\cos\theta,
\end{aligned}
\tag{4.10}
$$

the joint probability density function can be written by

$$
f(R, \dot{R}, \ddot{R}, \theta, \dot{\theta}, \ddot{\theta}) = \frac{R^3}{(2\pi)^3}\,\frac{1}{|A|}\, e^{-\frac{L}{2|A|}}
\tag{4.11}
$$

$$
0 \leq R < \infty,\quad -\infty < \dot{R} < \infty,\quad -\infty < \ddot{R} \leq 0,\quad 0 \leq \theta \leq 2\pi
$$

$$
-\infty < \dot{\theta} < \infty,\quad -\infty < \ddot{\theta} < \infty,
$$

and $$L = M_{11}R^2 + M_{22}(\dot{R}^2 + R^2 \dot{\theta}^2) + M_{33}(\ddot{R}^2 + 4\dot{R}^2 \dot{\theta}^2 + R^2 \dot{\theta}^4$$

$$+ R^2 \ddot{\theta}^2 - 2 R\ddot{R} \dot{\theta}^2 + 4R\dot{R} \dot{\theta} \ddot{\theta}) + 2 M_{12} R^2 \dot{\theta} + 2 M_{13}(R\ddot{R} - R^2 \dot{\theta}^2)$$

$$+2M_{23}(R\ddot{R} \dot{\theta} - 2\dot{R}^2 \dot{\theta} - R^2 \dot{\theta}^3 - R\dot{R} \ddot{\theta}) \tag{4.12}$$

By integrating Eq. (4.11) with respect to θ and $\ddot{\theta}$, we have

$$f(R, \dot{R}, \ddot{R}, \dot{\theta}) = \frac{R^2}{(2\pi)^{3/2}(|\mathbf{A}|\ M_{33})^{1/2}}\ e^{-\frac{P}{2|\mathbf{A}|}}, \tag{4.13}$$

$$0 \leq R < \infty, \quad -\infty < \dot{R} < \infty, \quad -\infty < \ddot{R} \leq 0, \quad -\infty < \dot{\theta} < \infty$$

where

$$P = M_{11}R^2 + M_{22}(\dot{R}^2 + R^2 \dot{\theta}^2) + M_{33}(\ddot{R}^2 + 4 \dot{R}^2 \dot{\theta}^2 + R^2 \dot{\theta}^4 - 2 R \ddot{R} \dot{\theta}^2)$$

$$+ 2M_{12}R^2 \dot{\theta} + 2 M_{13}(R \ddot{R} - R^2 \dot{\theta}^2) + 2M_{23}(R \ddot{R} \dot{\theta} - 2 \dot{R}^2 \dot{\theta} - R^2 \dot{\theta}^3)$$

$$+(\dot{R}^2/M_{33})(2 M_{33} \dot{\theta} + M_{23})^2. \tag{4.14}$$

Next, the probability density function of positive maxima of the envelope and phase velocity will be derived from Eq. (4.13). Let Ξ be a random variable representing the local positive maxima of the envelope. Then, the probability that Ξ will exceed a level ξ with a phase velocity $\dot{\theta}$ is equivalent to the expected value of the ratio of the number of positive maxima above the level ξ per unit time with velocity $\dot{\theta}$, denoted by $N(\xi,\dot{\theta})$, to the total number of positive maxima per unit time, denoted by $N(\xi_+)$. That is,

$$\Pr\{\Xi > \xi, \dot{\theta}\} = 1 - F(\xi, \dot{\theta}) = E\left[\frac{N(\xi,\dot{\theta})}{N(\xi_+)}\right] \tag{4.15}$$

where E[ ] stands for the expected value and $F(\xi,\dot{\theta})$ is the joint cumulative distribution function of ξ and $\dot{\theta}$.

If we assume that $N(\xi,\dot{\theta})/N(\xi_+)$ and $N(\xi_+)$ are statistically independent, then Eq. (4.15) can be written as

$$\Pr\{\Xi > \xi, \dot{\theta}\} = 1 - F(\xi, \dot{\theta}) = \frac{E[N(\xi,\dot{\theta})]}{E[N(\xi_+)]} \tag{4.16}$$

Taking $\dot{R} = 0$ at positive maxima into consideration, the numerator and denominator of Eq. (4.16) can be evaluated as follows:

$$E[N(\xi, \dot{\theta})] = \overline{N}_{\xi,\dot{\theta}} = -\int_{-\infty}^{\dot{\theta}} \int_{\xi}^{\infty} \int_{-\infty}^{0} \ddot{R}\, f(R, 0, \ddot{R}, \dot{\theta})\, d\ddot{R}\, dR\, d\dot{\theta} \tag{4.17}$$

$$E[N(\xi_+)] = \overline{N}_{\xi+} = -\int_{-\infty}^{\infty} \int_{0}^{\infty} \int_{-\infty}^{0} \ddot{R}\, f(R, 0, \ddot{R}, \dot{\theta})\, d\ddot{R}\, dR\, d\dot{\theta} \tag{4.18}$$

From Eqs. (4.16), (4.17), and (4.18), the joint probability density function of the envelope maxima and phase velocity $\dot{\theta}$ can be obtained as

$$f(\xi, \dot{\theta}) = \frac{\partial^2}{\partial\xi\, \partial\dot{\theta}} \left\{1 - \frac{\overline{N}_{\xi,\dot{\theta}}}{\overline{N}_{\xi+}}\right\} = \frac{\int_{-\infty}^{0} \ddot{R}\, f(\xi, 0, \ddot{R}, \dot{\theta})\, d\ddot{R}}{\int_{-\infty}^{\infty}\int_{0}^{\infty}\int_{-\infty}^{0} \ddot{R}\, f(R, 0, \ddot{R}, \dot{\theta})\, d\ddot{R}\, dR\, d\dot{\theta}}, \tag{4.19}$$

where $0 \leq \xi < \infty$, $-\infty < \dot{\theta} < \infty$.

The numerator of Eq. (4.19) can be evaluated analytically; however the integration with respect to R and $\dot{\theta}$ involved in the denominator can be evaluated numerically. The details of the derivation are given in Appendix J. The resulting joint probability density function of ξ and $\dot{\theta}$ becomes

$$f(\xi, \dot{\theta}) = \frac{\xi^2 e^{-\xi^2 u}[1 + \xi v\sqrt{\pi}\, e^{(\xi v)^2}\{1 + \Phi(\xi v)\}]}{\int_{-\infty}^{\infty}\int_0^{\infty} R^2 e^{-R^2 u}[1 + R v\sqrt{\pi}\, e^{(Rv)^2}\{1 + \Phi(Rv)\}]\, dR\, d\dot{\theta}},$$

$$0 < \xi < \infty, \qquad -\infty < \dot{\theta} < \infty \tag{4.20}$$

where

$$u = \frac{1}{2|\mathbf{A}|}\{M_{11} + 2 M_{12}\dot{\theta} + (M_{22} - 2 M_{13})\dot{\theta}^2 - 2 M_{23}\dot{\theta}^3 + M_{33}\dot{\theta}^4\}$$

$$v = \frac{1}{\{2|\mathbf{A}| M_{33}\}^{1/2}}(M_{13} + M_{23}\dot{\theta} - M_{33}\dot{\theta}^2) \tag{4.21}$$

$$\Phi(x) = \frac{2}{\sqrt{\pi}}\int_0^x e^{-t^2}\, dt.$$

From the joint probability density function of ξ and $\dot{\theta}$, we can derive the conditional probability density function of $\dot{\theta}$ given ξ. That is,

$$f(\dot{\theta}|\xi) = \frac{f(\xi, \dot{\theta})}{\int_{-\infty}^{\infty} f(\xi, \dot{\theta})\, d\dot{\theta}}. \tag{4.22}$$

Next, the conditional probability density function given in Eq. (4.22) will be converted to the conditional probability density function of the time interval between successive peaks above a specified level α by letting $\xi = \alpha$ and

$$\dot{\theta} = 2\pi k/\tau_\alpha, \tag{4.23}$$

where the parameter k is determined by the iteration method from a consideration of the probability of envelope crossing at the level α. The derivation of Eq. (4.23) is given in Appendix K along with a method to determine the parameter k. Thus, from Eqs. (4.20) through (4.23), the conditional probability density function of τ_α, given $\xi = \alpha$, can be written as follows:

$$f(\tau_\alpha|\alpha) = C \left. \frac{e^{-\xi^2 u}\,[1 + \xi\, v\sqrt{\pi}\; e^{(\xi v)^2}\{1 + \Phi(\xi v)\}]}{\int_{-\infty}^{\infty} e^{-\xi^2 u}[1 + \xi\, v\sqrt{\pi}\; e^{(\xi v)^2}\;\{1 + \Phi(\xi v)\}]\; d\dot{\theta}} \right|_{\substack{\xi = \alpha \\ \dot{\theta} = 2\pi k/\tau_\alpha}} \cdot \frac{2\pi k}{\tau_\alpha^2}$$

$$= C \frac{e^{-\alpha^2 u'}[1 + \alpha v'\sqrt{\pi}\; e^{(\alpha v')^2}\{1 + \Phi(\alpha v')\}]}{\int_{-\infty}^{\infty} e^{-\alpha^2 u}[1 + \alpha v\sqrt{\pi}\; e^{(\alpha v)^2}\{1 + \Phi(\alpha v)\}]\; d\dot{\theta}} \; \frac{2\pi k}{\tau_\alpha^2}, \tag{4.24}$$

where

$$u' = \frac{1}{2|\mathbf{A}|}\left[M_{11} + 2M_{12}\left(\frac{2\pi k}{\tau_\alpha}\right) + (M_{22} - 2M_{13})\left(\frac{2\pi k}{\tau_\alpha}\right)^2 - 2M_{23}\left(\frac{2\pi k}{\tau_\alpha}\right)^3 + M_{33}\left(\frac{2\pi k}{\tau_\alpha}\right)^4\right]$$

$$v' = \frac{1}{\{2|\mathbf{A}|\; M_{33}\}^{1/2}}\left\{M_{13} + M_{23}\left(\frac{2\pi k}{\tau_\alpha}\right) - M_{33}\left(\frac{2\pi k}{\tau_\alpha}\right)^2\right\} \tag{4.25}$$

C = constant for normalization of the probability density function

$= 1/\int_0^\infty f(\tau_\alpha|\alpha) \cdot d\tau_\alpha$.

We may recall the assumption that the time interval between successive peaks of the envelope above a specified level α is equal to the time interval between successive up-crossings of the level by the envelope. Hence, the conditional probability density function given in Eq. (4.24) represents, as well, the conditional probability density function of the time interval between successive up-crossings of a specified level α by the envelope.

Equation (4.24) can also be expressed in terms of the central moments defined in Eq. (4.5) with the aid of Eq. (4.9). In this case, the parameters u, u', v and v' become

$$u = \frac{1}{2|\mathbf{A}|}\{(\mu_2\,\mu_4 - \mu_3^2) + 2\mu_2\,\mu_3\,\dot{\theta} + (\mu_o\,\mu_4 - 3\,\mu_2^2)\,\dot{\theta}^2$$

$$- 2\mu_o\,\mu_3\,\dot{\theta}^3 + \mu_o\,\mu_2\,\dot{\theta}^4\}$$

$$u' = \frac{1}{2|\mathbf{A}|}\{(\mu_2\,\mu_4 - \mu_3^2) + 2\,\mu_2\,\mu_3\left(\frac{2\pi k}{\tau_\alpha}\right) + (\mu_o\,\mu_4 - 3\,\mu_2^2)\left(\frac{2\pi k}{\tau_\alpha}\right)^2$$

$$- 2\,\mu_o\,\mu_3\left(\frac{2\pi k}{\tau_\alpha}\right)^3 + \mu_o\,\mu_2\left(\frac{2\pi k}{\tau_\alpha}\right)^4\}$$

$$v = \left(\frac{\mu_o\,\mu_2}{2|\mathbf{A}|}\right)^{1/2}\{\frac{\mu_2}{\mu_o} + \frac{\mu_3}{\mu_o}\,\dot{\theta} - \dot{\theta}^2\}$$

$$v' = \left(\frac{\mu_o\,\mu_2}{2|\mathbf{A}|}\right)^{1/2}\{\frac{\mu_2}{\mu_o} + \frac{\mu_3}{\mu_o}\left(\frac{2\pi k}{\tau_\alpha}\right) - \left(\frac{2\pi k}{\tau_\alpha}\right)^2\} \qquad (4.26)$$

Numerical computations are carried out for the two-parameter wave spectrum with significant wave height of 10.0 m and a modal frequency of 0.42 rps shown in Figure 4.1. The probability density functions of the time interval between two successive envelope up-crossings for level of 5.0, 6.0, and 7.0 m above the still water level are computed by Eq. (4.24) and the results are shown in Figure 4.2.

4.2 Probability Distribution of Time Interval Between Wave Groups

As discussed in the Introduction referring to Figure 1.3, an up-crossing of the envelope does not necessarily mean the occurrence of a wave group. The probability of occurrence of wave groups when the envelope exceeds a specified level α was obtained in Eq. (3.40) of Chapter III. Let this probability be denoted by p. Then, it can be said that p-fraction of $f(\tau_\alpha|\alpha)$ derived in Eq. (4.24) represents the probability density function of the time interval between two wave groups, but (1-p)-fraction of $f(\tau_\alpha|\alpha)$ should be modified so that it is associated with the time interval between two wave groups. The modification of the probability density function may be made as follows:

First, let us consider three up-crossings of the envelope. If the first and third up-crossings are associated with wave groups but the time duration of the second up-crossing is so short that it does not constitute a wave group, then the time duration between two wave groups is that between the first and third up-crossings. This is the situation of the up-crossings A, B, and C shown in Figure 1.3. In this case the time interval between Points A and C is considered as the sum of two independent random variables, each having the probability density function derived in Eq. (4.24). By writing the sum of two independent time intervals as $\tau_{2\alpha}$, its probability density function for a specified level

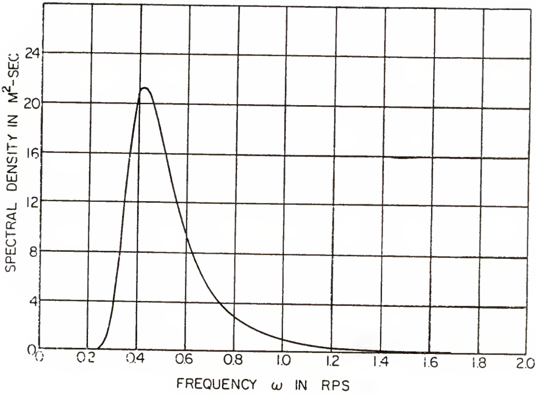


Figure 4.1 Two parameter wave spectrum (significant wave height 10.0 m., modal frquency 0.42 r.p.s.).

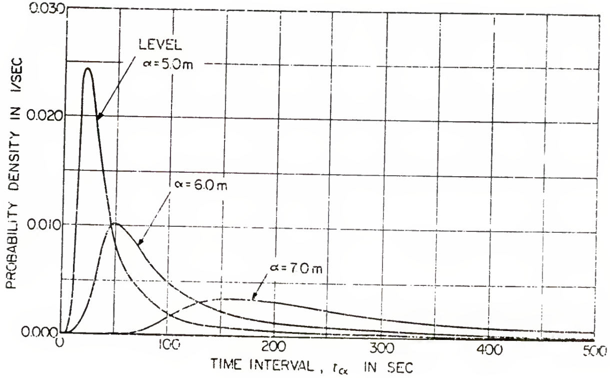


Figure 4.2 Probability density function of time interval between two successive envelope up-crossings for level of 5.0, 6.0, and 7.0 m. computed for the two parameter wave spectrum (significant wave height 10.0 m.).

α can be obtained by the following convolution integral:

$$f(\tau_{2\alpha}|\alpha) = \int_0^{\tau_{2\alpha}} f(\tau_\alpha|\alpha) \cdot f\left((\tau_{2\alpha} - \tau_\alpha)|\alpha\right) d\,\tau_\alpha \qquad (4.27)$$

where $f(\tau_\alpha|\alpha)$ = probability density function of time interval between successive up-crossings of envelope above a specified level α (see Eq. 4.24).

The same concept can be applied to four up-crossings of the envelope in which the first and fourth up-crossings are associated with wave groups. In this case the time interval between successive wave groups, denoted by $\tau_{3\alpha}$, can be considered as the sum of three independent random variables each having the probability density function given in Eq. (4.24). Its probability density function can be obtained from Eqs. (4.24) and (4.27) as

$$f(\tau_{3\alpha}|\alpha) = \int_0^{\tau_{3\alpha}} f(\tau_{2\alpha}|\alpha) \cdot f\left((\tau_{3\alpha} - \tau_{2\alpha})|\alpha\right) d\tau_{2\alpha} \qquad (4.28)$$

The same procedure can be extended to any number of up-crossings of the envelope in which the first and the last up-crossings are associated with wave groups. An example of the probability density function $f(\tau_{n\alpha}|\alpha)$ for n from 2 to 5, computed for the two-parameter wave spectrum, is shown in Figure 4.3. The crossing level α is 5 m in this example. Included also in the figure, for comparison, is the probability density function $f(\tau_\alpha|\alpha)$ shown in Figure 4.2.

From the results of analysis of wave records obtained at sea, it was found that, in practice, an upper limit of $\tau_{5\alpha}$ is sufficient for

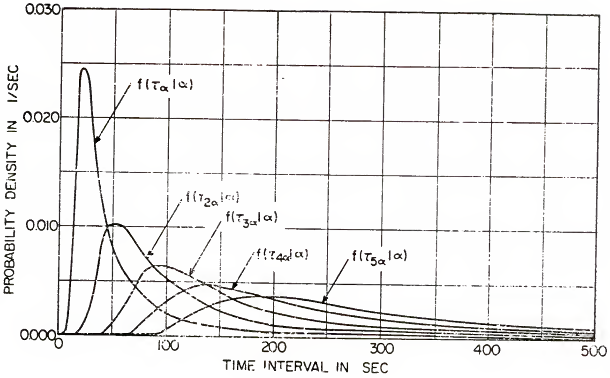


Figure 4.3 Probability density functions $f(\tau_{n\alpha}|\alpha)$ for n from 1 to 5 computed for the two parameter wave spectrum (significant wave height 10.0 m., level α = 5.0 m.)

consideration of the longest time interval between up-crossings for wave groups, and that $\tau_{2\alpha}$, $\tau_{3\alpha}$, $\tau_{4\alpha}$, and $\tau_{5\alpha}$ appear to have equal chances of occurrence for high level crossings. Here, $\tau_{5\alpha}$ implies the time interval between the first and the last up-crossings of a total of six crossings of the envelope. Therefore, in the present analysis, the probability density functions $f(\tau_{n\alpha}|\alpha)$ for n from 2 to 5 are evaluated through convolution integrals and then these probability density functions are accumulated with equal weight $(1-p)/4$ along with the probability density function $f(\tau_{\alpha}|\alpha)$ which has a weight p. Thus, in summary, the probability density function of the time interval between successive wave groups for a specified level α, denoted by $f(\tau_{\alpha G}|\alpha)$, can be written as

$$f(\tau_{\alpha G}|\alpha) = p \cdot f(\tau_{\alpha}|\alpha) + \frac{1-p}{4}\left\{ f(\tau_{2\alpha}|\alpha) + f(\tau_{3\alpha}|\alpha) + f(\tau_{4\alpha}|\alpha) + f(\tau_{5\alpha}|\alpha)\right\} \tag{4.29}$$

where p = probability of occurrence of wave groups when the envelope exceeds a specified level α (see Eq. (3.40) in Chapter 3).

$$f(\tau_{n\alpha}|\alpha) = \int_0^{\tau_{n\alpha}} f\left((\tau_{n\alpha} - \tau_{(n-1)\alpha})|\alpha\right) \cdot f\left(\tau_{(n-1)\alpha}|\alpha\right) \cdot d\tau_{(n-1)\alpha}$$

$$n = 2,\ 3,\ 4,\ \text{and}\ 5 \tag{4.30}$$

As an example, Figure 4.4 shows the probability density function $f(\tau_{\alpha G}|\alpha)$ computed for the two-parameter wave spectrum for the level $\alpha = 5$ m.

The average time interval between successive wave groups, denoted by $\bar{\tau}_{\alpha G}$, can be evaluated from Eq. (4.29) as

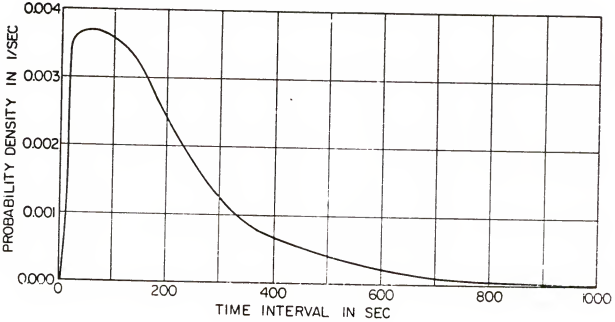


Figure 4.4 Probability density function of time interval between successive wave groups computed for the two parameter wave spectrum (significant wave height 10.0 m., level α = 5.0 m.)

$$\bar{\tau}_{\alpha G} = \int_0^\infty \tau_{\alpha G} \cdot f(\tau_{\alpha G}|\alpha)\, d\tau_{\alpha G} \tag{4.31}$$

4.3 Evaluation of Occurrence of Wave Groups

It is of considerable interest to evaluate for a given wave spectrum how many times a wave group will occur in a specified time period, say 30 minutes or one hour, as long as the sea severity is unchanged. For this, renewal theory is applied to the probability density function of the time interval between wave groups.

It has often been said that the wave group phenomenon obeys the Poisson random process. This assumption may be true if we consider the wave group phenomenon to be a rare event. In this case, the time interval between successive occurrence of the event (wave groups for the present problem) must follow the exponential probability law. However, as shown in Figure 4.4, the probability density function of the time interval between wave groups, $f(\tau_{\alpha G}|\alpha)$, cannot be considered to be an exponential distribution, in general. Hence, in the present analysis, the probability density function $f(\tau_{\alpha G}|\alpha)$ is approximated by the following gamma probability distribution which is a generalized form of the exponential distribution:

$$f(\tau_{\alpha G}) = \frac{1}{\Gamma(m)} \lambda^m \tau_{\alpha G}^{m-1} \cdot e^{-\lambda\tau_{\alpha G}} \qquad 0 < \tau_{\alpha G} < \infty \tag{4.32}$$

Note that $f(\tau_{\alpha G})$ reduces to the exponential distribution for $m = 1$. The parameters m and λ of the gamma distribution given in Eq. (4.32) can

be determined by equating the mean and variance computed from the probability density function given in Eq. (4.29) to the mean and variance of the gamma distribution. An example of a comparison between the probability density functions computed from Eq. (4.29) and the gamma distribution for the two-parameter wave spectrum is shown in Figure 4.5.

Based on the probability density function given in Eq. (4.32), the probability of n-occurrences of wave groups in a specified time period can be derived as follows:

Let T_n be the waiting time up to the n-th wave group. Then, we may write

$$T_n = (\tau_{\alpha G})_1 + (\tau_{\alpha G})_2 + (\tau_{\alpha G})_3 + \;\cdots\cdots\; + (\tau_{\alpha G})_n, \tag{4.33}$$

where all $\tau_{\alpha G}$ in Eq. (4.33) are statistically independent and follow the probability density function given in Eq. (4.32). Since the characteristic function of the gamma probability distribution given in Eq. (4.32) can be written as

$$\phi_{\tau_{\alpha G}}(u) = \left(1 - \frac{iu}{\lambda}\right)^{-m}, \tag{4.34}$$

the characteristic function of T_n becomes

$$\phi_{T_n}(u) = \left(1 - \frac{iu}{\lambda}\right)^{-mn}. \tag{4.35}$$

This is also the characteristic function of the gamma probability distribution with the parameters λ and mn. Thus, the probability density function of T_n can be obtained as

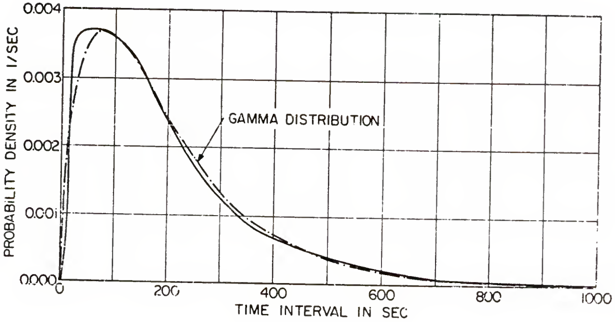


Figure 4.5 Comparison between probability density function of time interval between successive wave groups and gamma probability density function (the two-parameter wave spectrum of significant wave height 10.0 m., and level α = 5.0 m.).

$$f(T_n) = \frac{1}{\Gamma(mn)} \lambda^{mn} T_n^{mn-1} e^{-\lambda T_n}. \tag{4.36}$$

Similarly, the probability density function for the waiting time up to the (n+1)th wave group, denoted by T_{n+1}, can be written as

$$f(T_{n+1}) = \frac{1}{\Gamma\big(m(n+1)\big)} \lambda^{m(n+1)} T_n^{m(n+1)-1} e^{-\lambda T_{n+1}} \tag{4.37}$$

Then, we have the following relationship:

$$\Pr\left\{\begin{matrix}\text{n-occurrences of wave} \\ \text{groups in time t}\end{matrix}\right\} = \Pr\{T_n < t\} - \Pr\{T_{n+1} < t\}$$

$$= \int_0^t f(T_n)\, dT_n - \int_0^t f(T_{n+1})\, dT_{n+1}$$

$$= \frac{\gamma(mn, \lambda t)}{\Gamma(mn)} - \frac{\gamma\big(m(n+1), \lambda t\big)}{\Gamma\big(m(n+1)\big)}$$

$$n = 1, 2, 3, \ldots \tag{4.38}$$

where $\gamma(\ )$ = incomplete gamma function.

In particular, for n = 0, we have

$$\Pr\left\{\begin{matrix}\text{no occurrence of a} \\ \text{wave group in time t}\end{matrix}\right\} = 1 - \Pr\{T_1 < t\}$$

$$= 1 - \frac{\gamma(m, \lambda t)}{\Gamma(m)} \tag{4.39}$$

As an example of application of the prediction method derived in above, numerical computations are carried out using wave data measured in the North Sea off Norway. The recording time was 17 minutes, and the significant wave height was 8.13 m. The spectral density function obtained from data is shown as Wave Spectrum A in Figure 3.5 in Chapter III of the present dissertation.

The probability density function of the time interval between successive wave group, $f(\tau_{\alpha G})$, computed for a level of $\alpha = 4.0$ m is shown in Figure 4.6 along with the gamma distribution computed by Eq. (4.32).

The probabilities of wave groups in 17 minutes for a level of $\alpha = 4.0$ m are computed for various number of occurrences by Eq. (4.38) and the results are tabulated in Table 4.1 As can be seen in the table, the probability of occurrence of 4, 5, and 6 wave groups is high; specifically the highest possibility is 0.204 for 5 occurrences. This predicted result agrees very well with the measurements in which five wave groups were observed in 17 minutes at the level of 4 meters.

Currently available methods for evaluating the time interval between successive wave groups consider the average time interval between two successive envelope crossings of a specified level. This results in the average number of occurrences of wave groups being extremely large, the average being 37.7 occurrences in 17 minutes computed for the example shown in Table 4.1. The average number of occurrences of wave groups computed by the present theory is 5.5 in 17 minutes. This number can be obtained from the average time interval between successive wave groups given in Eq. (4.31).

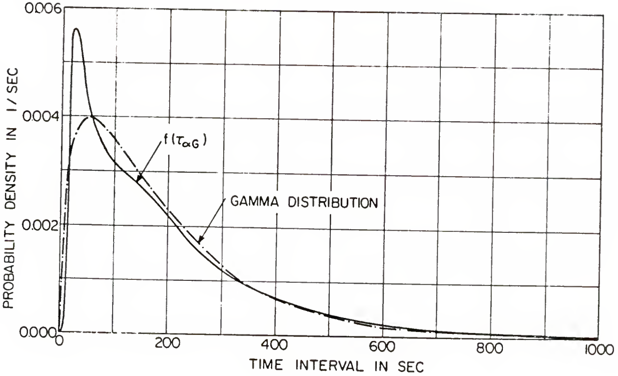


Figure 4.6 Comparison between probability density function of time interval between successive wave groups and gamma probability density function (measured spectrum of significant wave height 8.13 m., and level α = 4.0 m.)

Table 4.1 Predicted frequency of occurrence of wave groups and comparison with measured number of wave groups. Data measured in North Sea off Norway for 17 minutes, significant wave height 8.13 m., level $\alpha = 4.0$ m.

Predicted probability	Measured number of wave groups
Pr { No wave group } = 0.001 Pr { 1 wave group } = 0.012 Pr { 2 wave groups } = 0.050 Pr { 3 wave groups } = 0.116 Pr { 4 wave groups } = 0.180 Pr { 5 wave groups } = 0.204 Pr { 6 wave groups } = 0.178 Pr { 7 wave groups } = 0.126 Pr { 8 wave groups } = 0.073 Pr { 9 wave groups } = 0.036 Pr { more than 10 wave groups } = 0.024	5

CHAPTER V
CONCLUSIONS

This study presents a method for predicting various stochastic characteristics of wave groups in random seas where the water depth is sufficiently deep. The purpose of this study is to develop analytically, for a given wave spectrum, (i) the probability density function of the time duration of a wave group, (ii) the probability of occurrence of a specified number of waves in a wave group, (iii) the probability density function of the time interval between successive wave groups and (iv) the frequency of occurrence of wave groups in a specified time.

It has been customary, to date, to consider the exceedance of the wave envelope above a certain level to identify a wave group. However, if the time duration above a certain level is relatively short, there may be only one wave crest (or no wave crest) in the duration, which obviously does not constitute a wave group even though the envelope exceeds the specified level. Hence, in the present study, a sequence of (at least two) high waves exceeding a specified level is considered as a wave group.

The probability density function applicable for the time duration of the wave envelope above a certain level is analytically developed based on the concept associated with the level crossing problem of a narrow band Gaussian random process. This probability density function is then modified to that applicable for the wave group (Equation 3.41).

Formula to evaluate the probability of the number of wave crests in a group is also derived based on the probability density function of time duration associated with wave groups (Equation 3.48).

The probability density function of the time interval between successive envelope crossings of a specified level is developed, and later this probability density function is modified taking into consideration the probability of occurrence of wave groups when the envelope exceeds a specified level. The probability density function of the interval applicable for the wave groups, thusly derived, is given in Equation 4.29. Renewal theory is then applied to the probability density function of the time interval between wave groups for evaluating the frequency of occurrence of wave groups in a specified time, say 30 minutes or one hour (Equation 4.38).

The average time duration associated with the envelope exceeding a specified level computed by the newly developed theory agrees reasonably well with data observed in the North Sea for a high crossing level, while the average time duration computed by applying the formula in current use is substantially smaller than observed data (See Table 3.2). Also, the average time duration associated with wave groups computed by the present study agrees well with that which is observed in the data (Table 3.4).

The frequency of occurrence of wave groups predicted by the present theory agrees very well with that observed, while the frequency of occurrence computed by applying the formula currently used is substantially greater than observed data from the North Sea.

APPENDIX A DERIVATION OF ELEMENTS OF COVARIANCE MATRIX OF THE RANDOM VECTOR $\mathbf{X}' = (x_{c1}, \dot{x}_{c1}, x_{c2}, \dot{x}_{c2}, x_{s1}, \dot{x}_{s1}, x_{s2}, \dot{x}_{s2})$

The covariance matrix of the random vector $\mathbf{X}' = (x_{c1}, \dot{x}_{c1}, x_{c2}, \dot{x}_{c2}, x_{s1}, \dot{x}_{s1}, x_{s2}, \dot{x}_{s2})$ can be written by

$$\Sigma = \begin{pmatrix} E\,[x_{c1}^2] & E\,[x_{c1}\dot{x}_{c1}] & \cdots & E\,[x_{c1}\dot{x}_{s2}] \\ E\,[x_{c1}\dot{x}_{c1}] & E\,[\dot{x}_{c1}^2] & \cdots & E\,[\dot{x}_{c1}\dot{x}_{s2}] \\ \vdots & & & \vdots \\ E\,[x_{c1}\dot{x}_{s2}] & E\,[\dot{x}_{c1}\dot{x}_{s2}] & \cdots & E\,[\dot{x}_{s2}^2] \end{pmatrix} \tag{A.A.1}$$

where x_c, x_s are expressed by

$$x_c(t) = \sum_{n=1}^{\infty} c_n \cos\,[\,(\omega_n - \bar{\omega})t - \varepsilon_n]$$

$$x_s(t) = \sum_{n=1}^{\infty} c_n \sin\,[\,(\omega_n - \bar{\omega})t - \varepsilon_n] \tag{A.A.2}$$

The subscripts $i = 1,2$ of the elements of the covariance matrix are associated with the values of $x_c(t)$ and $x_s(t)$ at times t_1 and t_2 . ε_n are the phases. c_n are the amplitudes which are related with the spectral density function as

$$\frac{1}{2} c_n^2 = S(\omega_n) \cdot \Delta\omega_n \qquad \text{(A.A.3)}$$

Each element of the covariance matrix may be evaluated as follows:

$$E\,[x_{c1}^2] = E\left[\sum_{n=1}^{\infty} c_n \cos\{(\omega_n - \bar{\omega})t_1 - \varepsilon_n\} \sum_{n=1}^{\infty} c_n \cos\{(\omega_n - \bar{\omega})t_1 - \varepsilon_n\}\right]$$

$$= \sum_{n=1}^{\infty} c_n^2 \, \overline{\cos^2\{(\omega_n - \bar{\omega})t_1 - \varepsilon_n\}} = \sum_{n=1}^{\infty} \frac{1}{2} c_n^2$$

$$= \sum_{n=1}^{\infty} S(\omega_n) \cdot \Delta\omega_n$$

The bar denotes the average with respect to time t. By letting $\Delta\omega_n$ small, $E(x_{c1}^2)$ is equal to the area under the spectral density function denoted by μ_o. That is

$$E\,[x_{c1}^2] = \int_0^{\infty} S(\omega)d\omega = \mu_o \qquad \text{(A.A.4)}$$

$$E\,[x_{c1}\dot{x}_{c1}] = -E\left[\sum_{n=1}^{\infty} c_n \cos\{(\omega_n - \bar{\omega})t_1 - \varepsilon_n\} \sum_{n=1}^{\infty} c_n(\omega_n - \bar{\omega}) \sin\{(\omega_n - \bar{\omega})t_1 - \varepsilon_n\}\right]$$

$$= -\sum_{n=1}^{\infty} c_n^2(\omega_n - \bar{\omega}) \, \overline{\cos\{(\omega_n - \bar{\omega})t_1 - \varepsilon_n\} \sin\{(\omega_n - \bar{\omega})t_1 - \varepsilon_n\}} = 0 \qquad \text{(A.A.5)}$$

$$E\,[x_{c1}x_{c2}] = E\left[\sum_{n=1}^{\infty} c_n \cos\{(\omega_n - \bar{\omega})t_1 - \varepsilon_n\} \sum_{n=1}^{\infty} c_n \cos\{(\omega_n - \bar{\omega})(t_1 + \tau) - \varepsilon_n\}\right]$$

$$= \sum_{n=1}^{\infty} c_n^2 \,\overline{\cos^2\{(\omega_n - \bar{\omega})t_1 - \varepsilon_n\}\cos(\omega_n - \bar{\omega})\tau}$$

$$= \sum_{n=1}^{\infty} \frac{1}{2} c_n^2 \cos(\omega_n - \bar{\omega})\tau = \sum_{n=1}^{\infty} S(\omega_n)\cos(\omega_n - \bar{\omega})\tau \cdot \Delta\omega_n$$

Thus, $E\,[x_{c1}x_{c2}]$ is equal to the area under the function which is the product of spectral density function and $\cos(\omega - \bar{\omega})\tau$, and is denoted by ν_o. That is,

$$E\,[x_{c1}x_{c2}] = \int_0^{\infty} S(\omega)\cos(\omega - \bar{\omega})\tau d\omega = \nu_o \qquad \text{(A.A.6)}$$

$$E\,[x_{c1}\dot{x}_{c2}] = -E\left[\sum_{n=1}^{\infty} c_n \cos\{(\omega_n - \bar{\omega})t_1 - \varepsilon_n\}\right.$$

$$\left. \times \sum_{n=1}^{\infty} c_n(\omega_n - \bar{\omega}) \sin\{(\omega_n - \bar{\omega})(t_1 + \tau)\ \varepsilon_n\}\right]$$

$$= -\sum_{n=1}^{\infty} c_n^2 (\omega_n - \bar{\omega})\,\overline{\cos^2\{(\omega_n - \bar{\omega})t_1 - \varepsilon_n\}\sin(\omega_n - \bar{\omega})\tau}$$

$$= - \sum_{n=1}^{\infty} \frac{1}{2} c_n^2 (\omega_n - \bar{\omega}) \sin (\omega_n - \bar{\omega})\tau$$

$$= - \sum_{n=1}^{\infty} S(\omega_n) (\omega_n - \bar{\omega}) \sin (\omega_n - \bar{\omega})\tau \cdot \Delta\omega_n$$

Thus, $E[x_{c1}x'_{c2}]$ is equal to the derivative of ν_o with respect to τ, and it is given in the integral form as

$$E [x_{c1}x'_{c2}] = -\int_0^{\infty} S(\omega) (\omega - \bar{\omega}) \sin (\omega - \bar{\omega})\tau \, d\omega = -\nu_1 \qquad \text{(A.A.7)}$$

$$E [x_{c1}x_{s1}] = E\left[\sum_{n=1}^{\infty} c_n \cos \{(\omega_n - \bar{\omega})t_1 - \varepsilon_n\} \sum_{n=1}^{\infty} c_n \sin \{(\omega_n - \bar{\omega})t_1 - \varepsilon_n\}\right] = 0 \qquad \text{(A.A.8)}$$

$$E [x_{c1}x'_{s1}] = E\left[\sum_{n=1}^{\infty} c_n \cos \{(\omega_n - \bar{\omega})t_1 - \varepsilon_n\} \sum_{n=1}^{\infty} c_n (\omega_n - \bar{\omega}) \cos \{(\omega_n - \bar{\omega})t_1 - \varepsilon_n\}\right]$$

$$= \sum_{n=1}^{\infty} \frac{1}{2} c_n^2 (\omega_n - \bar{\omega}) = \sum_{n=1}^{\infty} S(\omega_n) (\omega_n - \bar{\omega}) \cdot \Delta\omega_n$$

Thus, $E [x_{c1}x'_{s1}]$ is equal to the first moment of the spectrum about its mean frequency and it becomes

$$E\,[x_{c1}\dot{x}_{s1}] = \int_0^\infty (\omega - \bar{\omega})\, S(\omega)\, d\omega = \mu_1 = 0 \tag{A.A.9}$$

$$E\,[x_{c1}x_{s2}] = E\left[\sum_{n=1}^{\infty} c_n \cos\,\{(\omega_n - \bar{\omega})t_1 - \varepsilon_n\} \sum_{n=1}^{\infty} c_n \sin\,\{(\omega_n - \bar{\omega})(t_1 + \tau) - \varepsilon_n\}\right]$$

$$= \sum_{n=1}^{\infty} \frac{1}{2}\, c_n^2 \sin\,(\omega_n - \bar{\omega})\tau = \sum_{n=1}^{\infty} S(\omega_n) \sin\,(\omega_n - \bar{\omega})\tau \cdot \Delta\omega_n$$

Thus, $E\,[x_{c1}x_{s2}]$ is equal to the area under the function which is the product of spectral density function and $\sin\,(\omega - \bar{\omega})\tau$, and is denoted by η_o. That is,

$$E\,[x_{c1}x_{s2}] = \int_0^\infty S(\omega) \sin\,(\omega - \bar{\omega})\tau\, d\omega = \eta_o \tag{A.A.10}$$

$$E\,[x_{c1}\dot{x}_{s2}] = E\left[\sum_{n=1}^{\infty} c_n \cos\,\{(\omega_n - \bar{\omega})t_1 - \varepsilon_n\} \sum_{n=1}^{\infty} c_n(\omega_n - \bar{\omega}) \cos\,\{(\omega_n - \bar{\omega})(t_1 + \tau) - \varepsilon_n\}\right]$$

$$= \sum_{n=1}^{\infty} \frac{1}{2}\, c_n^2\, (\omega_n - \bar{\omega}) \cdot \cos\,(\omega_n - \bar{\omega})\tau = \sum_{n=1}^{\infty} S(\omega_n)(\omega_n - \bar{\omega}) \cos\,(\omega_n - \bar{\omega})\tau \cdot \Delta\omega_n$$

Thus, $E\,[x_{c1}\dot{x}_{s2}]$ is the derivative of η_o with respect to τ. It is denoted by η_1 and is given in the integration form as

$$E\,[x_{c1}x'_{s2}] = \int_0^\infty (\omega - \bar{\omega})\, S(\omega) \cos(\omega - \bar{\omega})\tau \, d\omega = \eta_1 \tag{A.A.11}$$

$$E\,[x'^2_{c1}] = E\left[\sum_{n=1}^{\infty} c_n(\omega_n - \bar{\omega}) \sin\{(\omega_n - \bar{\omega})t_1 - \varepsilon_n\} \sum_{n=1}^{\infty} c_n(\omega_n - \bar{\omega}) \sin\{(\omega_n - \bar{\omega})t_1 - \varepsilon_n\}\right]$$

$$= \frac{1}{2} \sum_{n=1}^{\infty} (\omega_n - \bar{\omega})^2 c_n^2 = \sum_{n=1}^{\infty} S(\omega_n)\,(\omega_n - \bar{\omega})^2 \cdot \Delta\omega_n$$

Thus, $E\,[x'^2_{c1}]$ is equal to the second moment of the spectrum about its mean frequency which can be expressed by

$$E\,[x'^2_{c1}] = \int_0^\infty (\omega - \bar{\omega})^2\, S(\omega)\, d\omega = \mu_2 \tag{A.A.12}$$

$$E\,[x'_{c1}x_{c2}] = -\,E\left[\sum_{n=1}^{\infty} c_n \sin\{(\omega_n - \bar{\omega})t_1 - \varepsilon_n\}\,(\omega_n - \bar{\omega})\right.$$

$$\left. \times \sum_{n=1}^{\infty} c_n \cos\{(\omega_n - \bar{\omega})\,(t_1 + \tau) - \varepsilon_n\}\right]$$

$$= \sum_{n=1}^{\infty} \frac{1}{2}\, c_n^2\, (\omega_n - \bar{\omega}) \sin(\omega_n - \bar{\omega})\tau$$

$$= \sum_{n=1}^{\infty} S(\omega_n)\ (\omega_n - \bar{\omega})\ \sin\ (\omega_n - \bar{\omega})\tau \cdot \Delta\omega_n$$

Thus, $E\ [\dot{x}_{c1} x_{c2}]$ is equal to the derivative of ν_o with respect to τ but with opposite sign. It is given in the integral form as

$$E\ [\dot{x}_{c1} x_{c2}] = \int_0^{\infty} (\omega - \bar{\omega})\ S(\omega)\ \sin\ (\omega - \bar{\omega})\tau\ d\omega = \nu_1 \qquad \text{(A.A.13)}$$

$$E\ [\dot{x}_{c1} \dot{x}_{c2}] = E\left[\sum_{n=1}^{\infty} c_n(\omega_n - \bar{\omega})\ \sin\ \{(\omega_n - \bar{\omega})t_1 - \varepsilon_n\}\right.$$

$$\left. \times \sum_{n=1}^{\infty} c_n(\omega_n - \bar{\omega})\ \sin\ \{(\omega_n - \bar{\omega})(t_1 + \tau) - \varepsilon_n\}\right]$$

$$= \sum_{n=1}^{\infty} \frac{1}{2}\ c_n^2\ (\omega_n - \bar{\omega})^2 \cos\ \{(\omega_n - \bar{\omega})\ \tau\ \}$$

Thus, $E\ [\dot{x}_{c1} \dot{x}_{c2}]$ is equal to the derivative of ν_1 with respect to τ. It is given in the integral form as

$$E\ [\dot{x}_{c1} \dot{x}_{c2}] = \int_0^{\infty} (\omega - \bar{\omega})^2\ S(\omega)\ \cos\ (\omega - \bar{\omega})\tau\ d\omega = \nu_2 \qquad \text{(A.A.14)}$$

$$E\,[\dot{x}_{c1}x_{s1}] = -\,E\left[\sum_{n=1}^{\infty} c_n(\omega_n - \bar{\omega})\,\sin\,\{(\omega_n - \bar{\omega})t_1 - \varepsilon_n\}\sum_{n=1}^{\infty} c_n \sin\,\{(\omega_n - \bar{\omega})t_1 - \varepsilon_n\}\right]$$

$$= -\,\frac{1}{2}\sum_{n=1}^{\infty} c_n^2\,(\omega_n - \bar{\omega})$$

Thus, $E\,[\dot{x}_{c1}x_{s1}]$ is equal to the first moment of the spectral density function about its mean frequency, which is zero. That is,

$$E\,[\dot{x}_{c1}x_{s1}] = \int_0^{\infty} (\omega - \bar{\omega})\,S(\omega)\,d\omega = \mu_1 = 0 \qquad \text{(A.A.15)}$$

$$E\,[\dot{x}_{c1}\dot{x}_{s1}] = -\,E\left[\sum_{n=1}^{\infty} c_n(\omega_n - \bar{\omega})\,\sin\,\{(\omega_n - \bar{\omega})t_1 - \varepsilon_n\}\right.$$

$$\left. x\sum_{n=1}^{\infty} c_n(\omega_n - \bar{\omega})\,\cos\,\{(\omega_n - \bar{\omega})t_1 - \varepsilon_n\}\right]$$

$$= -\sum_{n=1}^{\infty} c_n^2(\omega_n - \bar{\omega})^2\,\overline{\sin\,\{(\omega_n - \bar{\omega})t_1 - \varepsilon_n\}\,\cos\,\{(\omega_n - \bar{\omega})t_1 - \varepsilon_n\}}$$

$$= 0 \qquad \text{(A.A.16)}$$

$$E\,[x_{c1}' x_{s2}] = -\,E\left[\sum_{n=1}^{\infty} c_n(\omega_n - \bar{\omega})\,\sin\{(\omega_n - \bar{\omega})t_1 - \varepsilon_n\} \sum_{n=1}^{\infty} c_n \sin\{(\omega_n - \bar{\omega})\,(t_1 + \tau) - \varepsilon_n\}\right]$$

$$= -\sum_{n=1}^{\infty} \frac{1}{2}\,c_n^2\,(\omega_n - \bar{\omega})\,\cos\,(\omega_n - \bar{\omega})\tau$$

Thus, $E\,[x_{c1}' x_{s2}]$ is equal to the derivative of η_o with respect to τ with opposite sign. Hence, it can be written as

$$E\,[x_{c1}' x_{s2}] = -\int_0^{\infty} (\omega - \bar{\omega})\,S(\omega)\,\cos\,(\omega - \bar{\omega})\tau\,d\omega = -\,\eta_1 \qquad \text{(A.A.17)}$$

$$E\,[x_{c1}' x_{s2}'] = -\,E\left[\sum_{n=1}^{\infty} c_n(\omega_n - \bar{\omega})\,\sin\,\{(\omega - \bar{\omega})t_1 - \varepsilon_n\}\right.$$

$$\left. \times \sum_{n=1}^{\infty} c_n(\omega_n - \bar{\omega})\,\cos\,\{(\omega_n - \bar{\omega})\,(t_1 + \tau) - \varepsilon_n\}\right]$$

$$= \frac{1}{2}\sum_{n=1}^{\infty} c_n^2\,(\omega_n - \bar{\omega})^2 \sin\,(\omega_n - \bar{\omega})\tau$$

Thus, $E\,[x_{c1}' x_{s2}']$ is equal to the derivative of $-\eta_1$ with respect to τ. That is,

$$E\,[x_{c1}' x_{s2}'] = \int_0^{\infty} (\omega - \bar{\omega})^2\,S(\omega)\,\sin\,(\omega - \bar{\omega})\tau\,d\omega = \eta_2 \qquad \text{(A.A.18)}$$

$$E\,[x_{c2}x'_{c2}] = -\,E\Bigg[\sum_{n=1}^{\infty} c_n \cos\,\{(\omega_n - \bar{\omega})(t_1 + \tau) - \varepsilon_n\}$$

$$\times \sum_{n=1}^{\infty} c_n(\omega_n - \bar{\omega})\,\sin\,\{(\omega_n - \bar{\omega})\,(t_1 + \tau) - \varepsilon_n\}\Bigg]$$

$$= -\sum_{n=1}^{\infty} c_n^2(\omega_n - \bar{\omega})\;\overline{\cos\{(\omega_n - \bar{\omega})(t_1 + \tau) - \varepsilon_n\}\,\sin\{(\omega_n - \bar{\omega})(t_1 + \tau) - \varepsilon_n\}}$$

$$= 0 \qquad \text{(A.A.19)}$$

$$E\,[x_{c2}x_{s1}] = E\Bigg[\sum_{n=1}^{\infty} c_n \cos\,\{(\omega_n - \bar{\omega})\,(t_1 + \tau) - \varepsilon_n\}\sum_{n=1}^{\infty} c_n \sin\,\{(\omega_n - \bar{\omega})t_1 - \varepsilon_n\}\Bigg]$$

$$= -\,\frac{1}{2}\sum_{n=1}^{\infty} c_n^2\,\sin\,(\omega_n - \bar{\omega})\tau$$

Thus, $E\,[x_{c2}x_{s1}]$ equal to η_o given (A.A.10) with opposite sign. That is,

$$E\,[x_{c2}x_{s1}] = -\int_0^{\infty} S(\omega)\,\sin\,(\omega - \bar{\omega})\tau\,d\omega = -\,\eta_o \qquad \text{(A.A.20)}$$

$$E[x_{c2}x'_{s1}] = E\left[\sum_{n=1}^{\infty} c_n \cos\{(\omega_n - \bar{\omega})(t_1 + \tau) - \varepsilon_n\}\right.$$

$$\left. \times \sum_{n=1}^{\infty} c_n(\omega_n - \bar{\omega}) \cos\{(\omega_n - \bar{\omega})t_1 - \varepsilon_n\}\right]$$

$$= \sum_{n=1}^{\infty} \frac{1}{2} c_n^2 (\omega_n - \bar{\omega}) \cos(\omega_n - \bar{\omega})\tau$$

Hence, $E[x_{c2}x'_{s1}]$ equal to η_1 given in (A.A.11). That is,

$$E[x_{c2}x'_{s1}] = \int_0^{\infty} (\omega - \bar{\omega}) S(\omega) \cos(\omega - \bar{\omega})\tau \, d\omega = \eta_1 \qquad \text{(A.A.21)}$$

$$E[x_{s2}x_{c2}] = E\left[\sum_{n=1}^{\infty} c_n \cos\{(\omega_n - \bar{\omega})(t_1 + \tau) - \varepsilon_n\} \sum_{n=1}^{\infty} c_n \sin\{(\omega_n - \bar{\omega})(t_1 + \tau) - \varepsilon_n\}\right]$$

$$= \sum_{n=1}^{\infty} c_n^2 \overline{\cos\{(\omega_n - \bar{\omega})(t_1 + \tau) - \varepsilon_n\} \sin\{(\omega_n - \bar{\omega})(t_1 + \tau) - \varepsilon_n\}} = 0$$

$$\text{(A.A.22)}$$

$$E[x_{c2}x'_{s2}] = E\left[\sum_{n=1}^{\infty} c_n \cos\{(\omega_n - \bar{\omega})(t_1 + \tau) - \varepsilon_n\}\right.$$

$$\times \sum_{n=1}^{\infty} c_n(\omega_n - \bar{\omega}) \cos \{(\omega_n - \bar{\omega}) (t_1 + \tau) - \varepsilon_n\}\Big]$$

$$= \frac{1}{2} \sum_{n=1}^{\infty} c_n^2(\omega_n - \bar{\omega})$$

Thus, $E\,[x_{c2}\dot{x}_{s2}]$ is equal to μ_1 which is zero. That is,

$$E\,[x_{c2}\dot{x}_{s2}] = \int_0^{\infty} (\omega - \bar{\omega})\ S(\omega)\ d\omega = \mu_1 = 0 \qquad \text{(A.A.23)}$$

$$E\,[\dot{x}_{c2}^2] = E\left[\left(\sum_{n=1}^{\infty} c_n(\omega_n - \bar{\omega}) \sin \{(\omega_n - \bar{\omega}) (t_1 + \tau) - \varepsilon_n\}\right)^2\right]$$

$$= \sum_{n=1}^{\infty} \frac{1}{2}\, c_n^2\ (\omega_n - \bar{\omega})^2$$

Thus, $E[\dot{x}_{c2}^2]$ is equal to the second moment of the spectral density function about its mean frequency, denoted by μ_2 . That is,

$$E\,[\dot{x}_{c2}^2] = \int_0^{\infty} (\omega - \bar{\omega})^2\ S(\omega)\ d\omega = \mu_2 \qquad \text{(A.A.24)}$$

$$E\,[x_{s1}\dot{x}_{c2}] = -\,E\Big[\sum_{n=1}^{\infty} c_n(\omega_n - \bar{\omega}) \sin \{(\omega_n - \bar{\omega}) (t_1 + \tau) - \varepsilon_n\}$$

$$\times \sum_{n=1}^{\infty} c_n \sin \{(\omega_n - \bar{\omega})\, t_1 - \varepsilon_n\}\Bigg]$$

$$= -\frac{1}{2}\sum_{n=1}^{\infty} c_n^2\, (\omega_n - \bar{\omega}) \cos\, (\omega_n - \bar{\omega})\tau$$

$$= -\sum_{n=1}^{\infty} (\omega_n - \bar{\omega})\, S(\omega_n) \cos\, (\omega_n - \bar{\omega})\tau \cdot \Delta\omega_n$$

Thus, $E[x_{s1}\dot{x}_{c2}]$ is equal to the derivative of η_o with respect of τ with opposite sign. That is,

$$E\, [x_{s1}\dot{x}_{c2}] = -\int_0^{\infty} (\omega - \bar{\omega})\, S(\omega) \cos\, (\omega - \bar{\omega})\tau\, d\omega = -\, \eta_1 \qquad \text{(A.A.25)}$$

$$E\, [\dot{x}_{c2}\dot{x}_{s1}] = -\, E\Bigg[\sum_{n=1}^{\infty} (\omega_n - \bar{\omega})\, c_n \sin\, \{(\omega_n - \bar{\omega})\, (t_1 + \tau) - \varepsilon_n\}$$

$$\times \sum_{n=1}^{\infty} c_n (\omega_n - \bar{\omega}) \cos\, \{(\omega_n - \bar{\omega}) t_1 - \varepsilon_n\}\Bigg]$$

$$= -\frac{1}{2}\sum_{n=1}^{\infty} c_n^2\, (\omega_n - \bar{\omega})^2 \sin\, (\omega_n - \bar{\omega})\tau$$

$$= - \sum_{n=1}^{\infty} S(\omega_n)\ (\omega_n - \bar{\omega})^2 \sin\ (\omega_n - \bar{\omega})\tau \cdot \Delta\omega_n$$

Thus, $E\ [x'_{c2} x'_{s1}]$ is equal to the second derivative of η_o with respect to τ. That is,

$$E\ [x'_{c2} x'_{s1}] = - \int_0^{\infty} (\omega - \bar{\omega})^2 S(\omega) \sin\ (\omega - \bar{\omega})\tau\ d\omega = - \eta_2 \tag{A.A.26}$$

$$E\ [x'_{c2} x_{s2}] = - E\Bigg[\sum_{n=1}^{\infty} c_n(\omega_n - \bar{\omega}) \sin\ \{(\omega_n - \bar{\omega})\ (t_1 + \tau) - \varepsilon_n\}$$

$$\times \sum_{n=1}^{\infty} c_n \sin\ \{(\omega_n - \bar{\omega})\ (t_1 + \tau) - \varepsilon_n\}\Bigg]$$

$$= - \sum_{n=1}^{\infty} c_n^2(\omega_n - \bar{\omega})\ \overline{\sin^2\ \{(\omega_n - \bar{\omega})\ (t_1 + \tau) - \varepsilon_n\}}$$

$$= - \sum_{n=1}^{\infty} \frac{1}{2}\ c_n^2\ (\omega_n - \bar{\omega}) = - \sum_{n=1}^{\infty} (\omega_n - \bar{\omega}) \cdot S(\omega_n) \cdot \Delta\omega_n$$

Thus, we may write

$$E\ [x'_{c2} x_{s2}] = - \int_0^{\infty} (\omega - \bar{\omega})\ S(\omega)\ d\omega = - \mu_1 = 0 \tag{A.A.27}$$

$$E\,[x'_{c2}x'_{s2}] = -\,E\left[\sum_{n=1}^{\infty} c_n(\omega_n - \bar{\omega})\,\sin\,\{(\omega_n - \bar{\omega})\,(t_1 + \tau) - \varepsilon_n\}\right.$$

$$\left.\times \sum_{n=1}^{\infty} c_n(\omega_n - \bar{\omega})\,\cos\,\{(\omega_n - \bar{\omega})\,(t_1 + \tau) - \varepsilon_n\}\right]$$

$$= -\sum_{n=1}^{\infty} c_n^2(\omega_n - \bar{\omega})^2\overline{\sin\{(\omega_n - \bar{\omega})(t_1 + \tau) - \varepsilon_n\}\cos\{(\omega_n - \bar{\omega})(t_1 + \tau) - \varepsilon_n\}}$$

$$= 0 \qquad \text{(A.A.28)}$$

$$E\,[x_{s1}^2] = E\left[\sum_{n=1}^{\infty} c_n \sin\,\{(\omega_n - \bar{\omega})\,t_1 - \varepsilon_n\}\right.$$

$$\left.\times \sum_{n=1}^{\infty} c_n \sin\,\{(\omega_n - \bar{\omega})t_1 - \varepsilon_n\}\right]$$

$$= \sum_{n=1}^{\infty} c_n^2\,\overline{\sin^2\,\{(\omega_n - \bar{\omega})t_1 - \varepsilon_n\}}$$

$$= \sum_{n=1}^{\infty} \frac{1}{2} \; c_n^2 = \sum_{n=1}^{\infty} S(\omega_n) \; \Delta\omega_n$$

Thus, we may write

$$E \; [x_{s1}^2] = \int_0^{\infty} S(\omega) \; d\omega = \mu_o \tag{A.A.29}$$

$$E \; [x_{s1}\dot{x}_{s1}] = E\left[\sum_{n=1}^{\infty} c_n \sin \{(\omega_n - \bar{\omega})t_1 - \varepsilon_n\}\right.$$

$$\left. \times \sum_{n=1}^{\infty} c_n(\omega_n - \bar{\omega}) \cos \{(\omega_n - \bar{\omega})t_1 - \varepsilon_n\}\right]$$

$$= \sum_{n=1}^{\infty} c_n^2 \; (\omega_n - \bar{\omega}) \; \overline{\sin \{(\omega_n - \bar{\omega})t_1 - \varepsilon_n\} \cos \{(\omega_n - \bar{\omega})t_1 - \varepsilon_n\}}$$

$$= 0 \tag{A.A.30}$$

$$E \; [x_{s1}x_{s2}] = E\left[\sum_{n=1}^{\infty} c_n \sin \{(\omega_n - \bar{\omega})t_1 - \varepsilon_n\}\right.$$

$$\times \sum_{n=1}^{\infty} c_n \sin \{(\omega_n - \bar{\omega}) (t_1 + \tau) - \varepsilon_n\}\Bigg]$$

$$= \frac{1}{2} \sum_{n=1}^{\infty} c_n^2 \cos (\omega_n - \bar{\omega})\tau = \sum_{n=1}^{\infty} S(\omega_n) \cos (\omega_n - \bar{\omega})\tau \, \Delta\omega_n$$

Thus, we have

$$E\,[x_{s1}x_{s2}] = \int_0^{\infty} S(\omega) \cos (\omega - \bar{\omega})\tau \, d\omega = \nu_o \qquad \text{(A.A.31)}$$

$$E\,[x_{s1}\dot{x}_{s2}] = -\,E\Bigg[\sum_{n=1}^{\infty} c_n \sin \{(\omega_n - \bar{\omega})t_1 - \varepsilon_n\}$$

$$\times \sum_{n=1}^{\infty} c_n(\omega_n - \bar{\omega}) \cos \{(\omega_n - \bar{\omega}) (t_1 + \tau) - \varepsilon_n\}\Bigg]$$

$$= -\sum_{n=1}^{\infty} \frac{1}{2} c_n^2 (\omega_n - \bar{\omega}) \sin (\omega_n - \bar{\omega})\tau$$

$$= -\sum_{n=1}^{\infty} S(\omega_n) (\omega_n - \bar{\omega}) \sin (\omega_n - \bar{\omega})\tau \, \Delta\omega_n$$

Thus, we have

$$E\,[x_{s1}\dot{x}_{s2}] = -\int_0^\infty S(\omega)\,(\omega-\bar{\omega})\,\sin\,(\omega-\bar{\omega})\tau\,d\omega = -\,\nu_1 \qquad \text{(A.A.32)}$$

$$E\,[\dot{x}_{s1}\dot{x}_{s2}] = E\Bigg[\sum_{n=1}^{\infty} c_n(\omega_n-\bar{\omega})\,\cos\,\{(\omega_n-\bar{\omega})t_1-\varepsilon_n\}$$

$$\times \sum_{n=1}^{\infty} c_n\,(\omega_n-\bar{\omega})\,\cos\,\{(\omega_n-\bar{\omega})\,(t_1+\tau)-\varepsilon_n\}\Bigg]$$

$$= \sum_{n=1}^{\infty} \frac{1}{2}\,c_n^2\,(\omega_n-\bar{\omega})^2 = \sum_{n=1}^{\infty} S(\omega_n)(\omega_n-\bar{\omega})^2\,\Delta\omega_n$$

$$= \int_0^\infty S(\omega)\,(\omega-\bar{\omega})^2\,d\omega = \mu_2 \qquad \text{(A.A.33)}$$

$$E\,[\dot{x}_{s1}x_{s2}] = E\Bigg[\sum_{n=1}^{\infty} c_n(\omega_n-\bar{\omega})\,\cos\,\{(\omega_n-\bar{\omega})t_1-\varepsilon_n\}$$

$$\times \sum_{n=1}^{\infty} c_n \sin\,\{(\omega_n-\bar{\omega})\,(t_1+\tau)-\varepsilon_n\}\Bigg]$$

$$= \sum_{n=1}^{\infty} \frac{1}{2} c_n^2 (\omega_n - \bar{\omega}) \sin (\omega_n - \bar{\omega})\tau$$

$$= \int_0^{\infty} S(\omega) (\omega - \bar{\omega}) \sin (\omega - \bar{\omega})\tau \, d\omega = \nu_1 \qquad \text{(A.A.34)}$$

$$E [x'_{s1} x'_{s2}] = E\left[\sum_{n=1}^{\infty} c_n(\omega_n - \bar{\omega}) \cos \{(\omega_n - \bar{\omega})t_1 - \varepsilon_n\}\right.$$

$$\left. \times \sum_{n=1}^{\infty} c_n(\omega_n - \bar{\omega}) \cos \{(\omega_n - \bar{\omega}) (t_1 + \tau) - \varepsilon_n\}\right]$$

$$= \sum_{n=1}^{\infty} \frac{1}{2} c_n^2 (\omega_n - \bar{\omega})^2 \cos (\omega_n - \bar{\omega})\tau$$

$$= \int_0^{\infty} S(\omega) (\omega - \bar{\omega})^2 \cos (\omega - \bar{\omega})\tau \, d\omega = \nu_2 \qquad \text{(A.A.35)}$$

Similarly the impressions below can be proved.

$$E [x_{s2}^2] = E\left[\sum_{n=1}^{\infty} c_n(\omega_n - \bar{\omega}) \cos \{(\omega_n - \bar{\omega}) (t_1 + \tau) - \varepsilon_n\}\right.$$

$$\times \sum_{n=1}^{\infty} c_n (\omega_n - \bar{\omega}) \cos \{(\omega_n - \bar{\omega}) (t_1 + \tau) - \varepsilon_n\}\Bigg]$$

$$= \sum_{n=1}^{\infty} \frac{1}{2} c_n^2 = \sum_{n=1}^{\infty} S(\omega_n) \Delta\omega_n = \int_0^{\infty} S(\omega) d\omega = \mu_o \qquad \text{(A.A.36)}$$

$$E [x_{s2} \dot{x}_{s2}] = E\Bigg[\sum_{n=1}^{\infty} c_n \sin \{(\omega_n - \bar{\omega}) (t_1 + \tau) - \varepsilon_n\}$$

$$\times \sum_{n=1}^{\infty} c_n (\omega_n - \bar{\omega}) \cos \{(\omega_n - \bar{\omega}) (t_1 + \tau) - \varepsilon_n\}\Bigg]$$

$$= 0 \qquad \text{(A.A.37)}$$

$$E [\dot{x}_{s2}^2] = E\Bigg[\sum_{n=1}^{\infty} c_n (\omega_n - \bar{\omega}) \cos \{(\omega_n - \bar{\omega})) (t_1 + \tau) - \varepsilon_n\}$$

$$\times \sum_{n=1}^{\infty} c_n (\omega_n - \bar{\omega}) \cos \{(\omega_n - \bar{\omega}) (t_1 + \tau) - \varepsilon_n\}\Bigg]$$

$$= \sum_{n=1}^{\infty} \frac{1}{2} c_n^2 (\omega_n - \bar{\omega})^2 = \sum_{n=1}^{\infty} S(\omega_n) (\omega_n - \bar{\omega})^2 \Delta\omega_n$$

$$= \int_0^{\infty} (\omega - \bar{\omega})^2 S(\omega) d\omega = \mu_2 \qquad \text{(A.A.38)}$$

APPENDIX B WAVE SPECTRUM COMPOSED OF TWO SPECTRA HAVING SYMMETRIC SHAPE

In order to simplify the covariance matrix given in Eq. (3.5), a given wave spectrum is decomposed into two parts; each being symmetric about its modal frequency as illustrated in Figure 3.1. One part contains primarily the lower frequency energy components of the wave spectrum, while the other contains the higher frequencies of the spectrum. For this, express each decomposed spectrum in the form of a normal probability distribution with mean ω_{mi} and variance σ_i^2. That is,

$$S_i(\omega) = \frac{m_{oi}}{\sqrt{2\pi}\,\sigma_i} \exp\left\{ - \frac{(\omega - \omega_{mi})^2}{2\sigma_i^2} \right\} \qquad i = 1, 2 \tag{A.B.1}$$

where, m_{oi} is the area under the spectrum so that the integration of Eq. (A.B.1) over the frequency domain is equal to m_{oi}. The mean value ω_{mi} is the modal frequency of $S_i(\omega)$. By letting the value of $S_i(\omega)$ at the modal frequency be a_i, the unknown variance σ_i^2 can be obtained as

$$\sigma_i^2 = \frac{1}{2\pi} \left(\frac{m_{oi}}{a_i} \right)^2 \tag{A.B.2}$$

Thus, the symmetric shape can be expressed by

$$S_i(\omega) = a_i\, e^{-\pi \left(\frac{a_i}{m_{oi}} \right)^2 (\omega - \omega_{mi})^2} \tag{A.B.3}$$

The second moment about the mean , μ_2, of the spectrum can be evaluated as

$$\mu_{2i} = \int_0^{\infty} (\omega - \omega_{mi})^2 \, S_i(\omega) \, d\omega = \frac{m_{oi}}{2\pi}\left(\frac{m_{oi}}{a_i}\right)^2 \qquad \text{(A.B.4)}$$

Two conditions are considered in representing the shape of an arbitrarily given spectrum by the sum of two spectra, $S_1(\omega)$ and $S_2(\omega)$, given in Eq. (A.B.3). These are:

(a) The area under the original spectrum, m_o, is equal to the sum of m_{o1} and m_{o2}, each pertaining to the area under the spectrum $S_1(\omega)$ and $S_2(\omega)$, respectively.

(b) Since the parameter μ_2 defined in Eq. (3.6) plays a significant role in developing the probability density function of time duration $\tau_{\alpha+}$, the sum of μ_{21} and μ_{22} evaluated for $S_1(\omega)$ and $S_2(\omega)$ respectively, is maintained constant and is equal to μ_2 of the original spectrum.

The procedure to determine the parameters of the two decomposed spectra is as follows:

(1) Divide the given spectrum $S(\omega)$ into two parts and evaluate the areas m_{o1} and m_{o2} (where $m_{o1} + m_{o2} = m_o$). Choose the modal frequencies ω_{m1} and ω_{m2}, and assume a_1 as the value of the spectrum at the modal frequency ω_{m1}. All these values are appropriately chosen and will be used as the initial values for the computations.

(2) Evaluate μ_{21} from Eq. (A.B.4) and obtain μ_{22} by subtracting μ_{21} from μ_2 which is calculated for the given spectrum.

(3) Evaluate a_2 from Eq. (A.B.4) for the thusly obtained μ_{22}.

(4) By applying a non-linear least squares fitting technique, repeat the above computation procedure for various values of m_{o1}, a_1, ω_1, and ω_{m2}, such that the difference between the shape of the original spectrum and the sum of two symmetric spectra, expressed by $\left| S(\omega) - \sum_{i=1}^{2} S_i(\omega) \right|^2$, is minimal.

APPENDIX C ELEMENTS OF COVARIANCE MATRIX FOR WAVES WITH SPECTRUM OF SYMMETRIC SHAPE

The covariance matrix of a random vector **X** associated with wave profile is given in Eq. (3.5). If the shape of a given wave spectrum is represented by the sum of two symmetric spectra, then the covariance matrix can be drastically simplified since all elements, η_o, η_1, and η_2 of the covariance matrix become zero. The proof is given in the below:

Since the shape of a given wave spectrum is represented by the sum of two symmetric spectra, the element of the covariance matrix can be written as the sum of two elements of the covariance matrices evaluated for each decomposed symmetric spectrum. For example, $\eta_o = \eta_{o1} + \eta_{o2}$, where η_{oi} (i = 1 and 2) is given by

$$\eta_{oi} = \int_0^\infty S_i(\omega) \sin (\omega - \omega_{mi})\tau \, d\omega \tag{A.C.1}$$

where $S_i(\omega)$ = spectrum with symmetric shape given in Eq. (3.7)

$$= a_i e^{-\pi\left(\frac{a_i}{m_{oi}}\right)^2 (\omega - \omega_{mi})^2}$$

ω_{mi} = modal frequency of $S_i(\omega)$

a_i = value of $S_i(\omega)$ at the modal frequency

m_{oi} = area under the spectrum $S_i(\omega)$

By letting

$$\pi\left(\frac{a_i}{m_{oi}}\right)^2 = b_i \quad \text{and} \quad \omega - \omega_{mi} = \sigma \tag{A.C.2}$$

we have

$$S_i(\sigma) = a_i e^{-b_i \sigma^2} \tag{A.C.3}$$

and thereby

$$\eta_{oi} = \int_{-\omega_{mi}}^{\infty} S_i(\sigma) \cdot \sin \tau\sigma \, d\sigma = a_i \int_{-\omega_{mi}}^{\infty} e^{-b_i \sigma^2} \sin \tau\sigma \cdot d\sigma \tag{A.C.4}$$

Since it is assumed that the symmetric spectrum $S_i(\sigma)$ is narrow-banded and its energy is concentrated in the neighborhood of its mean frequency, we may write η_{oi} as

$$\eta_{oi} = \int_{-\infty}^{\infty} S_i(\sigma) \cdot \sin \tau\sigma \, d\sigma = a_i \int_{-\infty}^{\infty} e^{-b_i \sigma^2} \cdot \sin \tau\sigma \cdot d\sigma \tag{A.C.5}$$

The integrand is the product of a symmetric and an unsymmetric functions, hence η_{oi} is equal to zero. Thus, it can be proved that $\eta_o = \eta_{o1} + \eta_{o2} = 0$.

In a similar fashion, η_{2i} defined as follows can be proved to be zero. That is,

$$\eta_{2i} = \int_{0}^{\infty} S_i(\omega) \cdot (\omega - \omega_{mi})^2 \sin (\omega - \omega_{mi})\tau \, d\omega$$

$$\sim a_i \int_{-\infty}^{\infty} e^{-b_i \sigma^2} \cdot \sigma^2 \sin \tau\sigma \cdot d\sigma = 0 \tag{A.C.6}$$

Thus, $\eta_2 = \eta_{21} + \eta_{22}$ becomes zero.

For η_{1i} we may write

$$\eta_{1i} = \int_0^\infty S_i(\omega) \cdot (\omega - \omega_{mi}) \cdot \cos(\omega - \omega_{mi})\tau \cdot d\omega$$

$$\sim a_i \int_{-\infty}^{\infty} \sigma \cdot e^{-b_i \sigma^2} \cos \tau\sigma \, d\sigma$$

$$= a_i \left\{ \left[-\frac{1}{2b_i} e^{-b_i \sigma^2} \cdot \cos \tau\sigma \right]_{-\infty}^{\infty} + \frac{\tau}{2b_i} \int_{-\infty}^{\infty} e^{-b_i \sigma^2} \sin \tau\sigma \, d\sigma \right\}$$

(A.C.7)

The first term of Eq. (A.C.7) becomes zero, and the second term is also equal to zero as shown in Eq. (A.C.5). Thus, η_{1i} and thereby η_1 becomes zero.

APPENDIX D DERIVATION OF JOINT PROBABILITY DENSITY FUNCTION $f(x_{c1}, \dot{x}_{c1}, x_{c2}, \dot{x}_{c2}, x_{s1}, \dot{x}_{s1}, x_{s2}, \dot{x}_{s2})$

Derivation of the joint probability density function of the random vector $\mathbf{X}' = (x_{c1}, \dot{x}_{c1}, x_{c2}, \dot{x}_{c2}, x_{s1}, \dot{x}_{s1}, x_{s2}, \dot{x}_{s2})$ given Eq. (3.9) is as follows: The joint normal probability density function of a n-dimensional random vector $\mathbf{X}$ can be written, in general, as

$$f(x_1, x_2, \dots x_n) = \frac{1}{(2\pi)^{n/2}} \frac{1}{|\Sigma|^{1/2}} \exp\{ -\frac{1}{2} (\mathbf{X} - \mu)' \, \Sigma^{-1} (\mathbf{X} - \mu)\} \quad \text{(A.D.1)}$$

where $\mathbf{X}' = (x_1, x_2, \dots\dots x_n)$

μ = vector of mean values

Σ = covariance matrix

$|\Sigma|$ = determinant of the covariance matrix Σ.

For the present problem, the covariance matrix of the random vector given in Eq. (3.5) is drastically simplified to that given in Eq. (3.8) by representing the shape of a given wave spectrum by the sum of two symmetric spectra. We may write each element of the covariance matrix of the random vector Σ to represent the sum of two elements given in Eq. (3.8) evaluated for each symmetric spectrum. Then, the covariance matrix Σ may be written as

$$\Sigma = \begin{pmatrix} A & 0 \\ 0 & A \end{pmatrix} \quad \text{(A.D.2)}$$

where

$$\mathbf{A} = \begin{pmatrix} \mu_o & 0 & \nu_o & -\nu_1 \\ 0 & \mu_2 & \nu_1 & \nu_2 \\ \nu_o & \nu_1 & \mu_o & 0 \\ -\nu_1 & \nu_2 & 0 & \mu_2 \end{pmatrix} \quad \text{(A.D.3)}$$

$$\mu_o = \mu_{o1} + \mu_{o2}$$
$$\nu_o = \nu_{o1} + \nu_{o2} \quad \text{etc.}$$

The inverse of the covariance matrix $\mathbf{\Sigma}$ can be obtained as

$$\mathbf{\Sigma}^{-1} = \begin{pmatrix} \mathbf{A}^{-1} & 0 \\ 0 & \mathbf{A}^{-1} \end{pmatrix} \quad \text{(A.D.4)}$$

From (A.D.2) we have $|\mathbf{\Sigma}| = |\mathbf{A}|^2$. Since the mean values of the random vector are zero the joint probability density function of $\mathbf{X}$ can be written following the expression given in Eq. (A.D.1) as

$$f(x_{c1}, \dot{x}_{c1}, \ldots\ldots x_{s2}, \dot{x}_{s2}) = \frac{1}{(2\pi)^4} \frac{1}{|\mathbf{A}|} e^{-\frac{K}{2}} \quad \text{(A.D.5)}$$

where $|\mathbf{A}| = \{(\mu_2 + \nu_2)(\mu_o - \nu_o) - \nu_1^2\}\{(\mu_2 - \nu_2)(\mu_o + \nu_o) - \nu_1^2\}$ (A.D.6)

$$K = (x_{c1}, \dot{x}_{c1} \ldots\ldots x_{s2}, \dot{x}_{s2}) \begin{pmatrix} \mathbf{A}^{-1} & 0 \\ 0 & \mathbf{A}^{-1} \end{pmatrix} \begin{pmatrix} x_{c1} \\ \dot{x}_{c1} \\ \vdots \\ x_{s2} \\ \dot{x}_{s2} \end{pmatrix}$$

$$= (x_{c1}, \overset{'}{x}_{c1}, x_{c2}, \overset{'}{x}_{c2}) \quad A^{-1} \begin{pmatrix} x_{c1} \\ \overset{'}{x}_{c1} \\ x_{c2} \\ \overset{'}{x}_{c2} \end{pmatrix} + (x_{s1}, \overset{'}{x}_{s1}, x_{s2}, \overset{'}{x}_{s2}) \quad A^{-1} \begin{pmatrix} x_{s1} \\ \overset{'}{x}_{s1} \\ x_{s2} \\ \overset{'}{x}_{s2} \end{pmatrix} \qquad (A.D.7)$$

and

$$A^{-1} = \frac{1}{A^{-1}} \begin{pmatrix} M_{11} & M_{12} & M_{13} & M_{14} \\ M_{12} & M_{22} & -M_{14} & M_{24} \\ M_{13} & -M_{14} & M_{11} & -M_{12} \\ M_{14} & M_{24} & -M_{12} & M_{22} \end{pmatrix} \qquad (A.D.8)$$

The elements M_{ij} in the above matrix can be evaluated from Eq. (A.D.3) as follows:

$$\begin{aligned} M_{11} &= \mu_o(\mu_2^2 - \nu_2^2) - \nu_1^2\mu_2 \\ M_{12} &= \nu_1(\nu_o\mu_2 - \nu_2\mu_o) \\ M_{13} &= -\nu_o(\mu_2^2 - \nu_2^2) + \nu_1^2\nu_2 \\ M_{14} &= \nu_1(\mu_o\mu_2 - \nu_1^2 - \nu_o\nu_2) \\ M_{22} &= \mu_2(\mu_o^2 - \nu_o^2) - \nu_1^2\,\mu_o \\ M_{24} &= -\nu_2(\mu_o^2 - \nu_o^2) + \nu_o\nu_1^2 \end{aligned} \qquad (A.D.9)$$

By using Eq. (A.D.8), K can be written as

$$K = \frac{1}{|A|} \quad M_{11}(x_{c1}^2 + x_{c2}^2 + x_{s1}^2 + x_{s2}^2) + M_{22}(x_{c1}'^2 + x_{c2}'^2 + x_{s1}'^2 + x_{s2}'^2)$$

$$+ 2M_{12}(x_{c1}x_{c1}' - x_{c2}x_{c2}' + x_{s1}x_{s1}' - x_{s2}x_{s2}') + 2M_{13}(x_{c1}x_{c2} + x_{s1}x_{s2})$$

$$+ 2M_{24}(x_{c1}'x_{c2}' + x_{s1}'x_{s2}') + 2M_{14}(x_{c1}x_{c2}' - x_{c2}x_{c1}' + x_{s1}x_{s2}' - x_{s2}x_{s1}') \tag{A.D.10}$$

APPENDIX E DERIVATION OF JOINT PROBABILITY DENSITY FUNCTION $f(\dot{R}_1, \dot{R}_2; \alpha)$

The joint probability density function of $\dot{R}_1$, $\dot{R}_2$, $\dot{\Theta}_1$, $\dot{\Theta}_2$ for a specified level α is given in Eq. (3.16). That is,

$$f(\dot{R}_1, \dot{R}_2, \dot{\Theta}_1, \dot{\Theta}_2; \alpha) = \frac{\alpha^4}{(2\pi)^3 \, |\mathbf{A}|} \int_0^{2\pi} e^{-P} d\phi \tag{A.E.1}$$

where

$$\begin{aligned} P = \frac{1}{2\,|\mathbf{A}|} [\, & 2\,(M_{11} + M_{13}\cos\phi)\,\alpha^2 + 2(M_{12} - M_{14}\cos\phi)\,\alpha\,(\dot{R}_1 - \dot{R}_2) \\ & + M_{22}(\dot{R}_1^2 + \dot{R}_2^2) + 2M_{24}\,\dot{R}_1\dot{R}_2\cos\phi + M_{22}\alpha^2(\dot{\Theta}_1^2 + \dot{\Theta}_2^2) \\ & + 2M_{24}\,\alpha^2\,\dot{\Theta}_1\dot{\Theta}_2\cos\phi + 2M_{24}\,\alpha\sin\phi\,(\dot{R}_1\dot{\Theta}_2 - \dot{R}_2\dot{\Theta}_1) \\ & + 2M_{14}\alpha^2(\dot{\Theta}_1 + \dot{\Theta}_2)\sin\phi\,] \end{aligned} \tag{A.E.2}$$

By integrating Eq. (A.E.1) with respect to $\dot{\Theta}_1$ and $\dot{\Theta}_2$, we have

$$f(\dot{R}_1, \dot{R}_2; \alpha) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(\dot{R}_1, \dot{R}_2, \dot{\Theta}_1, \dot{\Theta}_2; \alpha)\, d\dot{\Theta}_1 d\dot{\Theta}_2$$

$$= \frac{\alpha^2}{(2\pi)^2} \int_0^{2\pi} \frac{1}{(M_{22}^2 - M_{24}^2\cos^2\phi)^{1/2}} \exp\left\{ \frac{Q}{2\,|\mathbf{A}|\,(M_{22}^2 - M_{24}^2\cos^2\phi)} \right\} d\phi \tag{A.E.3}$$

where

$$Q = B_1 + B_2(\dot{R}_1 - \dot{R}_2) - M_{22}(M_{22}^2 - M_{24}^2)\,(\dot{R}_1^2 + \dot{R}_2^2) - 2\,M_{24}(M_{22}^2 - M_{24}^2)\,\dot{R}_1\dot{R}_2 \cos\phi \tag{A.E.4}$$

and

$$\begin{aligned} B_1 = -\,2\,\alpha^2 \Big[& M_{22}\,(M_{22}M_{11} - M_{14}^2) + (M_{22}^2 M_{13} + M_{14}^2\,M_{24}) \cos\phi \\ & - (M_{11}M_{24}^2 - M_{22}M_{14}^2) \cos^2\phi - M_{24}(M_{13}M_{24} + M_{14}^2) \cos^3\phi \Big] \\ B_2 = -\,2\,\alpha \Big[& M_{22}(M_{22}M_{12} - M_{24}M_{14}) - M_{14}(M_{22}^2 - M_{24}^2) \cos\phi \\ & - M_{24}(M_{12}M_{24} - M_{22}M_{14}) \cos^2\phi \Big] \end{aligned} \tag{A.E.5}$$

In order to evaluate the average number $\bar{N}_{\alpha+}(\tau_{\alpha+})$, which is the numerator of Eq. (3.1), the joint probability density function $f(\dot{R}_1, \dot{R}_2;\ \alpha)$ given in Eq. (A.E.3) must be further integrated with respect to $\dot{R}_1$ and $\dot{R}_2$. Since this integration is extremely difficult for the form presented in Eq. (A.E.3), we may apply the following technique used by Rice (1958) so that Eq. (A.E.3) will be reduced in a form feasible to carry out the integration with respect to $\dot{R}_1$ and $\dot{R}_2$. That is, change the random variables $\dot{R}_1$ and $\dot{R}_2$ to U_1 and U_2 by letting $\dot{R}_1 = U_1 - k_1$ and $\dot{R}_2 = U_2 - k_2$ and choose k_1 and k_2 such that the coefficients of U_1 and U_2 in the resulting expression of the term Q in Eq. (A.E.3) become zero. Thus, we have

$$B_2 + 2(M_{22}^2 - M_{24}^2)\,(k_1 M_{22} + k_2 M_{24}\cos\phi) = 0$$

(A.E.6)

$$- B_2 + 2(M_{22}^2 - M_{24}^2)\,(k_2 M_{22} + k_1 M_{24}\cos\phi) = 0$$

From the above equation, k_1 and k_2 can be determined as

$$k_1 = - k_2 = \frac{\alpha\left[(M_{12}M_{22} - M_{14}M_{24}) + (M_{12}M_{24} - M_{14}M_{22})\cos\phi\right]}{M_{22}^2 - M_{24}^2} \qquad \text{(A.E.7)}$$

The following relationships obtained from Eq. (A.E.9) are also used in simplifying Eqs. (A.E.4) and (A.E.5):

$$M_{11}M_{22} - M_{14}^2 = \{\mu_2(\mu_o^2 - \nu_o^2) - \nu_1^2\mu_o\}\,\{\mu_o(\mu_2^2 - \nu_2^2) - \nu_1^2\mu_2\} - \nu_1^2(\mu_o\mu_2 - \nu_1^2 - \nu_o\nu_2)^2$$

$$= (\mu_o\mu_2 - \nu_1^2)\,|\mathbf{A}| \qquad \text{(A.E.8)}$$

where $|\mathbf{A}|$ is given in Eq. (A.E.6). That is,

$$|\mathbf{A}| = \{(\mu_2 + \nu_2)\,(\mu_o - \nu_o) - \nu_1^2\}\{(\mu_2 - \nu_2)\,(\mu_o + \nu_o) - \nu_1^2\}$$

$$M_{13}M_{24} + M_{14}^2 = (\nu_2^2\,\nu_o + \nu_2\nu_1^2 - \mu_2^2\nu_o)\,(\nu_o^2\nu_2 + \nu_o\nu_1^2 - \mu_o^2\nu_2) + \nu_1^2(\mu_o\mu_2 - \nu_1^2 - \nu_o\nu_2)^2$$

$$= (\nu_o\nu_2 + \nu_1^2)\,|\mathbf{A}| \qquad \text{(A.E.9)}$$

$$M_{12}M_{22} - M_{14}M_{24} = \{\mu_2(\mu_o^2 - \nu_o^2) - \nu_1^2\mu_o\}\,\nu_1(\nu_o\mu_2 - \nu_2\mu_o)$$

$$- (\nu_o^2\nu_2 + \nu_o\nu_1^2 - \mu_o^2\nu_2)\ \nu_1(\mu_o\mu_2 - \nu_1^2 - \nu_o\nu_2) = \nu_1\nu_o\ |\mathbf{A}| \qquad \text{(A.E.10)}$$

$$M_{22}^2 - M_{24}^2 = \{\mu_2(\mu_o^2 - \nu_o^2) - \nu_1^2\mu_o\}^2 - (\nu_o^2\nu_2 + \nu_o\nu_1^2 - \nu_2\mu_o^2)^2$$

$$= (\mu_o^2 - \nu_o^2)\ |\mathbf{A}| \qquad \text{(A.E.11)}$$

$$M_{12}M_{24} - M_{14}M_{22} = \nu_1(\nu_o\mu_2 - \nu_2\mu_o)\ (\nu_o\nu_1^2 + \nu_o^2\ \nu_2 - \mu_o^2\nu_2)$$

$$- \nu_1\{\mu_2(\mu_o^2 - \nu_o^2) - \nu_1^2\ \mu_o\}\ (\mu_o\mu_2 - \nu_1^2 - \nu_o\nu_2) = - \mu_o\nu_1\ |\mathbf{A}|$$

$$\text{(A.E.12)}$$

$$M_{13}M_{22}^2 + M_{14}^2M_{24} = M_{13}(M_{22}^2 - M_{24}^2) + M_{24}(M_{13}M_{24} + M_{14}^2) \qquad \text{(A.E.13)}$$

$$M_{11}M_{24}^2 - M_{14}^2M_{22} = - M_{11}(M_{22}^2 - M_{24}^2) + M_{22}(M_{14}M_{22} - M_{14}^2) \qquad \text{(A.E.14)}$$

Let us write

$$\rho = \frac{M_{24}\cos\phi}{M_{22}} = \frac{-\nu_2(\mu_o^2 - \nu_o^2) + \nu_1^2\ \nu_o}{\mu_2(\mu_o^2 - \nu_o^2) - \nu_1^2\ \nu_o}\cos\phi \qquad \text{(A.E.15)}$$

where $|\rho| \leqq 1$ following the definition of M_{22} and M_{24}.

Then, from Eqs. (A.E.7) through (A.E.15), Q given in Eq. (A.E.4) can be written in terms of U_1, and U_2 as

$$Q = - (\mu_o^2 - \nu_o^2)\ M_{22}(U_1^2 + U_2^2 + 2\rho\ U_1U_2) - \frac{2\alpha^2 M_{22}}{\mu_o^2 - \nu_o^2}\ (1 - \rho^2)\ (\mu_o - \nu_o\cos\phi)$$

$$\text{(A.E.16)}$$

where, the random variables (U_1, U_2) and (R_1', R_2') have the following relationships:

$$U_1 = R_1' - k, \qquad U_2 = R_2' + k$$

$$k = -k_1 = k_2 = \frac{\alpha \nu_1 (\mu_o \cos\phi - \nu_o)}{\mu_o^2 - \nu_o^2} \tag{A.E.17}$$

From Eq. (A.E.17), Q given in Eq. (A.E.16) can be expressed in terms of R_1', and R_2', and thereby the joint probability density function $f(R_1', R_2'; \alpha)$ can now be written as

$$f(R_1', R_2'; \alpha) = \frac{\alpha^2}{(2\pi)^2} \int_0^{2\pi} \frac{1}{M_{22}(1-\rho^2)^{1/2}} \exp\left\{ - \left[\frac{\alpha^2(\mu_o - \nu_o \cos\phi)}{\mu_o^2 - \nu_o^2} + \frac{\mu_o^2 - \nu_o^2}{2M_{22}(1-\rho^2)} \right.\right.$$

$$\left.\left. \times \{R_1'^2 + R_2'^2 + 2\rho R_1' R_2' - 2k(1-\rho)(R_1' - R_2' - k)\} \right] \right\} d\phi \tag{A.E.18}$$

$$0 \leqq R_1' < \infty \quad \text{and} \quad -\infty < R_2' \leqq 0$$

APPENDIX F SERIES EXPANSION OF BIVARIATE NORMAL DISTRIBUTION

The joint probability density function of two normally distributed random variables is given, in general, by

$$f(x,y) = \frac{1}{2\pi\,\sigma_1\sigma_2\,(1-\rho^2)^{1/2}} \times \exp\left\{-\frac{1}{2(1-\rho^2)}\left[\frac{(x-\mu_1)^2}{\sigma_1^2} - 2\rho\,\frac{(x-\mu_1)\,(y-\mu_2)}{\sigma_1\,\sigma_2} + \frac{(y-\mu_2)^2}{\sigma_2^2}\right]\right\}$$

(A.F.1)

where, μ_1, μ_2 are the mean values and σ_1^2, σ_2^2 are the variances of x and y, respectively. ρ is the correlation coefficient between x and y. For $\mu_1 = \mu_2 = 0$, the joint probability density function becomes

$$f(x,y) = \frac{1}{2\pi\,\sigma_1\sigma_2(1-\rho^2)^{1/2}} \exp\left\{-\frac{1}{2(1-\rho^2)}\left[\left(\frac{x}{\sigma_1}\right)^2 - 2\rho\left(\frac{x}{\sigma_1}\right)\left(\frac{y}{\sigma_2}\right) + \left(\frac{y}{\sigma_2}\right)^2\right]\right\}$$

(A.F.2)

Cramér (1946) shows the series expansion of the joint probability density function given in Eq. (A.F.2) as follows:

$$f(x_1y) = \frac{1}{\sigma_1\sigma_2}\sum_{n=0}^{\infty} \rho^n \frac{\Phi^{(n+1)}\left(\frac{x}{\sigma_1}\right) \cdot \Phi^{(n+1)}\left(\frac{y}{\sigma_2}\right)}{n!}$$

$$= \frac{1}{\sigma_1\sigma_2}\left\{\Phi^{(1)}\left(\frac{x}{\sigma_1}\right) \cdot \Phi^{(1)}\left(\frac{y}{\sigma_2}\right) + \rho\,\Phi^{(2)}\left(\frac{x}{\sigma_1}\right) \cdot \Phi^{(2)}\left(\frac{y}{\sigma_2}\right) + \ldots\ldots\right\}$$

(A.F.3)

where, $\Phi(\ )$ is the standardized normal cumulative distribution function. For instance, $\Phi(\frac{x}{\sigma_1})$ is given as

$$\Phi(\frac{x}{\sigma_1}) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x/\sigma_1} \exp \{ - \frac{t^2}{2} \} \, dt \tag{A.F.4}$$

By taking the first term of the expansion given in Eq. (A.F.3) and by letting $\sigma_1 = \sigma_2 = 1$, we have

$$\frac{1}{2\pi(1-\rho^2)^{1/2}} \exp \left\{ - \frac{1}{2(1-\rho^2)} (x^2 - 2\rho xy + y^2) \right\} \sim \frac{1}{2\pi} e^{- \frac{x^2 + y^2}{2}} \tag{A.F.5}$$

that is,

$$e^{- \frac{1}{2(1-\rho^2)} (x^2 - 2\rho xy + y^2)} \sim (1-\rho^2)^{1/2} \, e^{- \frac{x^2 + y^2}{2}} \tag{A.F.6}$$

This is the approximate formula given in Eq. (3.22).

APPENDIX G DERIVATION OF EXPECTED NUMBER $\bar{N}_{\alpha-}(\tau_{\alpha-})$

The expected number (per unit time) of the envelope downward crossing with velocity $\dot{R}_1$ followed by the upward crossing with velocity $\dot{R}_2$, denoted by $\bar{N}_{\alpha-}(\tau_{\alpha-})$, can be written by

$$\bar{N}_{\alpha-}(\tau_{\alpha-}) = -\int_0^{\infty}\int_{-\infty}^{0} \dot{R}_1\dot{R}_2 f(\dot{R}_1, \dot{R}_2; \alpha)\, d\dot{R}_1 d\dot{R}_2 \qquad \begin{array}{l} -\infty < \dot{R}_1 \leq 0 \\ 0 \leq \dot{R}_2 < \infty \end{array} \tag{A.G.1}$$

The procedure to obtain $\bar{N}_{\alpha-}(\tau_{\alpha-})$ is the same as that for the derivation of the expected number $\bar{N}_{\alpha+}(\tau_{\alpha+})$ shown in Eqs. (3.2) through (3.29). However, the difference takes place in the transformation of random variables from $(\dot{R}_1, \dot{R}_2)$ to (U, V) given in Eq. (3.20). That is, for the derivation of $\bar{N}_{\alpha-}(\tau_{\alpha-})$, the following transformation is made:

$$\begin{aligned} \dot{R}_1 &= -\sqrt{\frac{M_{22}}{\mu_o^2 - \nu_o^2}}\, U = -\sqrt{\mu_2 - \frac{\mu_o \nu_1^2}{\mu_o^2 - \nu_o^2}}\, U \\ \dot{R}_2 &= \sqrt{\frac{M_{22}}{\mu_o^2 - \nu_o^2}}\, V = \sqrt{\mu_2 - \frac{\mu_o \nu_1^2}{\mu_o^2 - \nu_o^2}}\, V \end{aligned} \tag{A.G.2}$$

Then, the joint probability density function of the random variables U and V for a specified level α becomes

$$f(u, v; \alpha) = \left(\frac{\alpha}{2\pi}\right)^2 \frac{1}{\mu_o^2 - \nu_o^2} \int_0^{2\pi} \frac{1}{\sqrt{1-\rho^2}}$$

$$\times \exp\left\{ - \left[\frac{\alpha^2(\mu_o - \nu_o \cos \phi)}{\mu_o^2 - \nu_o^2} + \frac{1}{2(1-\rho^2)} (u^2 - 2\rho\, u\, v + v^2) \right.\right.$$

$$\left.\left. - \frac{1}{1+\rho} \left\{ (u + v)\left(k \sqrt{\frac{\mu_o^2 - \nu_o^2}{M_{22}}} \right) + \left(k \sqrt{\frac{\mu_o^2 - \nu_o^2}{M_{22}}} \right)^2 \right\} \right] \right\} d\phi$$

$$0 \leqq u < \infty \text{ and } 0 \leqq v < \infty \qquad \text{(A.G.3)}$$

By applying the property associated with the bi-variate normal probability distribution given in Appendix F to the 2nd term of the exponential expression in Eq. (A.G.2), the joint probability density function $f(u, v; \alpha)$ approximately becomes

$$f(u, v; \alpha) = \left(\frac{\alpha}{2\pi}\right)^2 \frac{1}{\mu_o^2 - \nu_o^2} \int_o^{2\pi} \exp\left\{ - \left[\frac{\alpha^2(\mu_2 - \nu_o \cos \phi)}{\mu_o^2 - \nu_o^2} + \frac{u^2 + v^2}{2} \right.\right.$$

$$\left.\left. + \frac{1}{1+\rho} \left\{ (u + v)\left(k \sqrt{\frac{\mu_o^2 - \nu_o^2}{M_{22}}} \right) + \left(k \sqrt{\frac{\mu_o^2 - \nu_o^2}{M_{22}}} \right)^2 \right\} \right] \right\} d\phi \qquad \text{(A.G.4)}$$

By transformation of the random variables $\dot{R}_1$, and $\dot{R}_2$ to U and V, respectively, Eq. (A.G.1) can be expressed in terms of U and V as follows:

$$\bar{N}_{\alpha-}(\tau_{\alpha-}) = \frac{M_{22}}{\mu_o^2 - \nu_o^2} \int_o^{\infty}\int_o^{\infty} u\, v\, f(u, v; \alpha)\, du\, dv \qquad \text{(A.G.5)}$$

Then, from Eqs. (A.G.3) and (A.G.4), we have

$$\bar{N}_{\alpha-}(\tau_{\alpha-}) = \left(\frac{\alpha}{2\pi}\right)^2 \frac{M_{22}}{(\mu_o^2 - \nu_o^2)^2}$$

$$\times \int_0^{2\pi} \left(\int_0^{\infty} \int_0^{\infty} u\ v\ \exp\left\{ - \left[\frac{u^2 + v^2}{2} + (u + v) \frac{k}{1+\rho} \sqrt{\frac{\mu_o^2 - \nu_o^2}{M_{22}}} \right] \right\} du\ dv \right)$$

$$\times \exp\left\{ - \left[\frac{\alpha^2(\mu_o - \nu_o \cos\phi)}{\mu_o^2 - \nu_o^2} + \frac{1}{1+\rho}\left(k \sqrt{\frac{\mu_o^2 - \nu_o^2}{M_{22}}} \right)^2 \right] \right\} d\phi \tag{A.G.6}$$

Since the terms associated with U and V are symmetric we can write

$$\int_0^{\infty} \int_0^{\infty} u\ v\ \exp\left\{ - \left[\frac{u^2 + v^2}{2} + (u + v) \frac{k}{1+\rho} \sqrt{\frac{\mu_o^2 - \nu_o^2}{M_{22}}} \right] \right\} du\ dv$$

$$= \left(\int_0^{\infty} u\ \exp\left\{ - \left[\frac{u^2}{2} + u \frac{k}{1+\rho} \sqrt{\frac{\mu_o^2 - \nu_o^2}{M_{22}}} \right] \right\} du \right)^2 \tag{A.G.7}$$

By letting

$$- \frac{k}{1+\rho} \sqrt{\frac{\mu_o^2 - \nu_o^2}{M_{22}}} = 2\gamma \tag{A.G.8}$$

the integration given in Eq. (A.G.7) can be obtained as

$$\int_0^{\infty} u\ e^{-(\frac{u}{2} - 2\gamma u)}\ du = 1 - \sqrt{2\pi}\ \gamma\ e^{2\gamma^2}\left\{1 - \Phi(\sqrt{2}\ \gamma)\right\}$$

where $$\Phi(z) = \frac{2}{\sqrt{\pi}} \int_0^z e^{-t^2} dt \tag{A.G.9}$$

Hence, $\bar{N}_{\alpha-}(\tau_{\alpha-})$ can be evaluated by

$$\bar{N}_{\alpha-}(\tau_{\alpha-}) = \frac{\alpha^2}{2\pi} \frac{M_{22}}{(\mu_o^2 - \nu_o^2)^2} \int_o^{2\pi} \exp\left\{ - \left[\frac{\alpha^2(\mu_o - \nu_o \cos\phi)}{\mu_o^2 - \nu_o^2} + 4(1+\rho) \gamma^2 \right] \right\}$$

$$\times \left(1 - \sqrt{2\pi}\ \gamma\, e^{2\gamma^2} \left\{ 1 - \Phi(\sqrt{2}\ \gamma) \right\} \right)^2 d\phi$$

$$= \frac{\alpha^2}{2\pi} \frac{M_{22}}{(\mu_o^2 - \nu_o^2)^2} \int_o^{2\pi} \exp\left\{ - \left[\frac{\alpha^2(\mu_o - \nu_o \cos\phi)}{\mu_o^2 - \nu_o^2} + 4\rho\gamma^2 \right] \right\}$$

$$\times \left(e^{-2\gamma^2} - \sqrt{2\pi}\ \gamma \left\{ 1 - \Phi(\sqrt{2}\ \gamma) \right\} \right)^2 d\phi \tag{A.G.10}$$

APPENDIX H DERIVATION OF ELEMENTS OF COVARIANCE MATRIX OF THE RANDOM VECTOR $X' = (x_c, \dot{x}_c, \ddot{x}_c, x_s, \dot{x}_s, \ddot{x}_s)$

The covariance matrix of the random vector $X' = (x_c, \dot{x}_c, \ddot{x}_c, x_s, \dot{x}_s, \ddot{x}_s)$ can be written by

$$\Sigma = \begin{pmatrix} E[x_c^2] & E[x_c \dot{x}_c] & \cdots & E[x_c \ddot{x}_s] \\ E[\dot{x}_c x_c] & E[\dot{x}_c^2] & \cdots & E[\dot{x}_c \ddot{x}_s] \\ \vdots & & & \vdots \\ E[\ddot{x}_s x_c] & E[\ddot{x}_s \dot{x}_c] & \cdots & E[\ddot{x}_s^2] \end{pmatrix}, \tag{A.H.1}$$

where x_c, x_s are expressed by

$$x_c(t) = \sum_{n=1}^{\infty} c_n \cos[(\omega_n - \bar{\omega})t - \varepsilon_n]$$
$$x_s(t) = \sum_{n=1}^{\infty} c_n \sin[(\omega_n - \bar{\omega})t - \varepsilon_n]. \tag{A.H.2}$$

ε_n are the phases. c_n are the amplitudes which are related with the spectral density function by

$$\frac{1}{2} c_n^2 = S(\omega_n)\, \Delta\omega_n. \tag{A.H.3}$$

In Appendix A of this study, the elements associated with displacement and velocity were evaluated. Those are

$$E[x_c^{\,2}] = \mu_0,\ E[x_c\, x_c'] = 0,\ \ E[x_c\, x_s] = 0,\ \ E[x_c\, x_s'] = 0,$$

$$E[x_c'^2] = \mu_2,\ \ E[x_c'\, x_s] = 0,\ \ E[x_s^2] = \mu_0,\ \ E[x_s\, x_s'] = 0 \qquad \text{(A.H.4)}$$

and

$$E[x_s'^2] = \mu_2 \,.$$

The derivation of elements of the covariance matrix associated with acceleration is given in this Appendix as follows:

$$E[x_c x_c''] = -E\left[\sum_{n=1}^{\infty} c_n \cos\{(\omega_n - \overline{\omega})t - \varepsilon_n\} \sum_{n=1}^{\infty} c_n(\omega_n - \overline{\omega})^2 \cos\{(\omega_n - \overline{\omega})t - \varepsilon_n\}\right]$$

$$= -\sum_{n=1}^{\infty} c_n^2(\omega_n - \overline{\omega})^2\ \overline{\cos^2\{(\omega_n - \overline{\omega})t - \varepsilon_n\}}$$

$$= -\sum_{n=1}^{\infty} \frac{1}{2} c_n^2(\omega_n - \overline{\omega})^2 = -\sum_{n=1}^{\infty} (\omega_n - \overline{\omega})^2\ S(\omega_n)\ \Delta\omega_n$$

The bar denotes the average with respect to time t. By letting $\Delta\omega_n$ small, $E[x_c\, x_c'']$ is equal to the second moment of the spectrum about its mean frequency and it becomes

$$E[x_c\, x_c''] = -\int_0^{\infty} (\omega - \overline{\omega})^2\ S(\omega)\ d\omega = -\ \mu_2 \qquad \text{(A.H.5)}$$

$$E[x_c x_s''] = -E\left[\sum_{n=1}^{\infty} c_n \cos\{(\omega_n - \overline{\omega})t - \varepsilon_n\} \cdot \sum_{n=1}^{\infty} c_n(\omega_n - \overline{\omega})^2 \sin\{(\omega_n - \overline{\omega})t - \varepsilon_n\}\right]$$

$$= 0 \qquad \text{(A.H.6)}$$

$$E[x_c' x_c''] = -E\left[\sum_{n=1}^{\infty} c_n(\omega_n - \bar{\omega}) \sin\{(\omega_n - \bar{\omega})t - \varepsilon_n\}\right.$$

$$\left. \times \sum_{n=1}^{\infty} c_n(\omega_n - \bar{\omega})^2 \cos\{(\omega_n - \bar{\omega})t - \varepsilon_n\}\right]$$

$$= - \sum_{n=1}^{\infty} c_n^2(\omega_n - \bar{\omega})^3 \overline{\cos\{(\omega_n - \bar{\omega})t - \varepsilon_n\} \sin\{(\omega_n - \bar{\omega})t - \varepsilon_n\}}$$

$$= 0 \qquad \text{(A.H.7)}$$

$$E[x_c' x_s''] = E\left[\sum_{n=1}^{\infty} c_n(\omega_n - \bar{\omega}) \sin\{(\omega_n - \bar{\omega})t - \varepsilon_n\}\right.$$

$$\left. \times \sum_{n=1}^{\infty} c_n(\omega_n - \bar{\omega})^2 \sin\{(\omega_n - \bar{\omega})t - \varepsilon_n\}\right]$$

$$= \sum_{n=1}^{\infty} c_n^2(\omega_n - \bar{\omega})^3 \overline{\sin^2\{(\omega_n - \bar{\omega})t - \varepsilon_n\}} = \sum_{n=1}^{\infty} \frac{1}{2} c_n^2(\omega_n - \bar{\omega})^3$$

$$= \sum_{n=1}^{\infty} (\omega_n - \bar{\omega})^3 S(\omega_n) \Delta\omega_n .$$

Thus, $E[x_c' x_s'']$ is equal to the third moment of spectrum about its mean frequency which can be expressed by

$$E[x_c' x_s''] = \int_0^{\infty} (\omega - \bar{\omega})^3 S(\omega) d\omega = \mu_3 \qquad \text{(A.H.8)}$$

$$E[x_c''^2] = E\left[\sum_{n=1}^{\infty} c_n(\omega_n - \bar{\omega})^2 \cos\{(\omega_n - \bar{\omega})t - \varepsilon_n\}\right.$$

$$\times \sum_{n=1}^{\infty} c_n(\omega_n - \overline{\omega})^2 \cos\{(\omega_n - \overline{\omega})t - \varepsilon_n\}\Big]$$

$$= \sum_{n=1}^{\infty} \frac{1}{2} c_n^2(\omega_n - \overline{\omega})^4 = \sum_{n=1}^{\infty} (\omega_n - \overline{\omega})^4 S(\omega_n) \Delta\omega_n .$$

Thus, $E[\ddot{x}_c^2]$ is equal to the fourth moment of the spectral density function about its mean frequency. That is,

$$E[\ddot{x}_c^2] = \int_0^{\infty} (\omega - \overline{\omega})^4 S(\omega)\, d\omega = \mu_4 . \qquad \text{(A.H.9)}$$

$$E[\ddot{x}_c \ddot{x}_s] = E\Big[\sum_{n=1}^{\infty} c_n(\omega_n - \overline{\omega})^2 \cos\{(\omega_n - \overline{\omega})t - \varepsilon_n\}$$

$$\times \sum_{n=1}^{\infty} c_n(\omega_n - \overline{\omega})^2 \sin\{(\omega_n - \overline{\omega})t - \varepsilon_n\}\Big] = 0 \qquad \text{(A.H.10)}$$

$$E[x_s \ddot{x}_s] = -E\Big[\sum_{n=1}^{\infty} c_n \sin\{(\omega_n - \overline{\omega})t - \varepsilon_n\}$$

$$\times \sum_{n=1}^{\infty} c_n(\omega_n - \overline{\omega})^2 \sin\{(\omega_n - \overline{\omega})t - \varepsilon_n\}\Big]$$

$$= - \sum_{n=1}^{\infty} c_n^2(\omega_n - \overline{\omega})^2 \overline{\sin^2\{(\omega_n - \overline{\omega})t - \varepsilon_n\}}$$

$$= - \sum_{n=1}^{\infty} \frac{1}{2} c_n^2(\omega_n - \overline{\omega})^2 = - \sum_{n=1}^{\infty} (\omega_n - \overline{\omega})^2 S(\omega_n) \Delta\omega_n .$$

Thus, $E[x_s\ddot{x}_s]$ is equal to the negative second moment of the spectral density function about its mean frequency. That is,

$$E[x_s \ddot{x}_s] = -\int_0^{\infty} (\omega - \bar{\omega})^2 \, S(\omega) \, d\omega = -\mu_2 \qquad \text{(A.H.11)}$$

$$E(\dot{x}_s \ddot{x}_s) = E\Big[\sum_{n=1}^{\infty} c_n(\omega_n - \bar{\omega})^2 \cos\{(\omega_n - \bar{\omega})t - \varepsilon_n\}$$

$$\times \sum_{n=1}^{\infty} c_n(\omega_n - \bar{\omega})^2 \sin\{(\omega_n - \bar{\omega})t - \varepsilon_n\}\Big] = 0 \qquad \text{(A.H.12)}$$

$$E[\ddot{x}_s^2] = E\Big[\sum_{n=1}^{\infty} c_n(\omega_n - \bar{\omega})^2 \sin\{(\omega_n - \bar{\omega})t - \varepsilon_n\}$$

$$\times \sum_{n=1}^{\infty} c_n(\omega_n - \bar{\omega})^2 \sin\{(\omega_n - \bar{\omega})t - \varepsilon_n\}\Big]$$

$$= \sum_{n=1}^{\infty} c_n^2(\omega_n - \bar{\omega})^4 \, \overline{\sin^2\{(\omega_n - \bar{\omega})t - \varepsilon_n\}}$$

$$= \sum_{n=1}^{\infty} \frac{1}{2} c_n^2(\omega_n - \bar{\omega})^4 = \sum_{n=1}^{\infty} (\omega_n - \bar{\omega})^4 \, S(\omega_n) \, \Delta\omega_n$$

Thus, $E[\ddot{x}_s^2]$ is equal to the fourth moment of the spectral density function about its mean frequency which is denoted by

$$E[\ddot{x}_s^2] = \int_0^{\infty} (\omega - \bar{\omega})^4 \, S(\omega) \, d\omega = \mu_4 \qquad \text{(A.H.13)}$$

APPENDIX I DERIVATION OF JOINT PROBABILITY DENSITY FUNCTION

$$f(x_c, \dot{x}_c, \ddot{x}_c, x_s, \dot{x}_s, \ddot{x}_s)$$

The derivation of the joint probability density function of the random vector $\mathbf{X'} = (x_c, \dot{x}_c, \ddot{x}_c, x_s, \dot{x}_s, \ddot{x}_s)$ given in Eq. (4.7) is as follows:

The joint normal probability density function of a 6-dimensional random vector $\mathbf{X}$ with zero mean can be written as

$$f(x_1, x_2, \dots x_6) = \frac{1}{(2\pi)^3} \frac{1}{|\Sigma|^{1/2}} \exp\left\{- \frac{1}{2} \mathbf{X}' \Sigma^{-1} \mathbf{X}\right\} \tag{A.I.1}$$

where $\mathbf{X}' = (x_1, x_2, \dots x_6)$

Σ = covariance matrix

$|\Sigma|$ = determinant of the covariance matrix

For the present problem, the covariance matrix Σ is given in Eq. (4.6), and we can write

$$\Sigma = \begin{pmatrix} \mathbf{A} & 0 \\ \mathbf{0} & \mathbf{B} \end{pmatrix} \tag{A.I.2}$$

where

$$\mathbf{A} = \begin{pmatrix} \mu_0 & 0 & -\mu_2 \\ 0 & \mu_2 & -\mu_3 \\ -\mu_2 & -\mu_3 & \mu_4 \end{pmatrix}, \qquad \mathbf{B} = \begin{pmatrix} \mu_0 & 0 & -\mu_2 \\ 0 & \mu_2 & \mu_3 \\ -\mu_2 & \mu_3 & \mu_4 \end{pmatrix}. \tag{A.I.3}$$

The inverse of the covariance matrix Σ can be obtained as

$$\Sigma^{-1} = \begin{pmatrix} A^{-1} & 0 \\ 0 & B^{-1} \end{pmatrix} \tag{A.I.4}$$

Since the determinants of the two submatrices are equal, we can write $|\Sigma| = |A|^2$. Hence, the joint probability density function of X can be written following the expression given in Eq. (A.I.1) as

$$f(x_c, \dot{x}_s, \ddot{x}_c, x_s, \dot{x}_c, \ddot{x}_s) = \frac{1}{(2\pi)^3} \cdot \frac{1}{|A|} \cdot e^{-\frac{K}{2}} \tag{A.I.5}$$

where, $|A| = |B| = \mu_o(\mu_2\,\mu_4 - \mu_3^2) - \mu_2^3$ (A.I.6)

$$K = (x_c, \dot{x}_s, \ldots \ddot{x}_s) \begin{pmatrix} A^{-1} & 0 \\ 0 & B^{-1} \end{pmatrix} \begin{pmatrix} x_c \\ \dot{x}_s \\ \vdots \\ \ddot{x}_s \end{pmatrix}$$

$$= (x_c, \dot{x}_s, \ddot{x}_c)\; A^{-1} \begin{pmatrix} x_c \\ \dot{x}_s \\ \ddot{x}_c \end{pmatrix} + (x_s, \dot{x}_c, \ddot{x}_s)\; B^{-1} \begin{pmatrix} x_s \\ \dot{x}_c \\ \ddot{x}_s \end{pmatrix} \tag{A.I.7}$$

A^{-1} and B^{-1} are given as follows:

$$A^{-1} = \frac{1}{|A|} \begin{pmatrix} M_{11} & M_{12} & M_{13} \\ M_{12} & M_{22} & M_{23} \\ M_{13} & M_{23} & M_{33} \end{pmatrix}$$

$$B^{-1} = \frac{1}{|A|}\begin{pmatrix} M_{11} & -M_{12} & M_{13} \\ -M_{12} & M_{22} & -M_{23} \\ M_{13} & -M_{23} & M_{33} \end{pmatrix} \tag{A.I.8}$$

The elements M_{ij} in the above matrices can be evaluated from Eq. (A.I.3) as follows:

$$\begin{aligned} M_{11} &= \mu_2 \, \mu_4 - \mu_3^2 \\ M_{22} &= \mu_o \, \mu_4 - \mu_2^2 \\ M_{33} &= \mu_o \, \mu_2 \\ M_{12} &= \mu_2 \, \mu_3 \\ M_{13} &= \mu_2^2 \\ M_{23} &= \mu_o \, \mu_3 \end{aligned} \tag{A.I.9}$$

From Eqs. (A.I.7) through (A.I.9), we can write K as

$$K = \frac{1}{|A|} \, [M_{11}(x_c^2 + x_s^2) + M_{22}(x_c'^2 + x_s'^2) + M_{33}(x_c''^2 + x_s''^2)$$

$$+ 2\,M_{12}(x_c \, x_s' - x_c' \, x_s) + 2\,M_{13}(x_c \, x_c'' + x_s \, x_s'') + 2\,M_{33}(x_c'' \, x_s' - x_c' \, x_s'')]. \tag{A.I.10}$$

APPENDIX J DERIVATION OF JOINT PROBABILITY DENSITY FUNCTION $f(\xi, \dot{\theta})$

The joint probability density function of R, $\dot{R}$, $\ddot{R}$, $\dot{\theta}$ is given by Eq. (4.13). That is,

$$f(R, \dot{R}, \ddot{R}, \dot{\theta}) = \frac{R^2}{(2\pi)^{3/2}(|\mathbf{A}| M_{33})^{1/2}} e^{-\frac{P}{2|\mathbf{A}|}} \tag{A.J.1}$$

where

$$\begin{aligned} P = {} & M_{11}R^2 + M_{22}(\dot{R}^2 + R^2 \dot{\theta}^2) + M_{33}(\ddot{R}^2 + 4 \dot{R}^2 \dot{\theta}^2 - \dot{R}^2 \dot{\theta}^4 \\ & -2 R \ddot{R} \dot{\theta}^2) + 2 M_{12} R^2 \dot{\theta} + 2 M_{13}(R \ddot{R} - R^2 \dot{\theta}^2) \\ & + 2 M_{23}(R \ddot{R} \dot{\theta} - 2 \dot{R}^2 \dot{\theta} - R^2 \dot{\theta}^3) \\ & +(\dot{R}^2/M_{33})(2 M_{33} \dot{\theta} + M_{23})^2 \end{aligned} \tag{A.J.2}$$

By multiplying Eq. (A.J.1) by the delta function $\delta(\dot{R})$ and by integrating the product with respect to $\dot{R}$, we have

$$\begin{aligned} f(R, 0, \ddot{R}, \dot{\theta}) &= \int_{-\infty}^{+\infty} \delta(\dot{R})\, f(R, \dot{R}, \ddot{R}, \dot{\theta})\, d\dot{R} \\ &= \frac{R^2}{(2\pi)^{3/2}(|\mathbf{A}|\ M_{33})^{1/2}} e^{-\frac{P'}{2|\mathbf{A}|}} \end{aligned} \tag{A.J.3}$$

where

$$\begin{aligned} P' = {} & M_{11} R^2 + M_{22} R^2 \dot{\theta}^2 + M_{33}(\ddot{R}^2 + R^2 \dot{\theta}^4 - 2 R \ddot{R} \dot{\theta}^2) \\ & + 2 M_{12} R^2 \dot{\theta} + 2 M_{13}(R \ddot{R} - R^2 \dot{\theta}^2) \end{aligned}$$

$$+ 2 M_{23}(R \ddot{R} \dot{\theta} - R^2 \dot{\theta}^3)$$

$$0 < R < \infty, \quad -\infty < \ddot{R} \leq 0, \quad -\infty < \dot{\theta} < \infty \tag{A.J.4}$$

By using the joint probability density function given in Eq. (A.J.3), we can evaluate the joint probability density function $f(\xi,\dot{\theta})$ given in Eq. (4.19) which is repeated here; namely,

$$f(\xi,\dot{\theta}) = \frac{\partial^2}{\partial\xi\,\partial\dot{\theta}} \left\{1 - \frac{\overline{N}_{\xi,\dot{\theta}}}{\overline{N}_{\xi+}}\right\} = \frac{\int_{-\infty}^{0} \ddot{R} f(\xi, 0, \ddot{R}, \dot{\theta})\, d\ddot{R}}{\int_{-\infty}^{+\infty}\int_{0}^{\infty}\int_{-\infty}^{0} \ddot{R} f(R, 0, \ddot{R}, \dot{\theta}) d\ddot{R}\, dR\, d\dot{\theta}}$$

$$0 < \xi < \infty, \quad -\infty < \dot{\theta} < \infty$$

For evaluating the numerator of the above equation, we will transform the random variable $\ddot{R}$ to H defined as

$$\ddot{R} = -\left(\frac{2|A|}{M_{33}}\right)^{1/2} \cdot H \tag{A.J.5}$$

Then, we can express the numerator as

$$\int_{-\infty}^{0} \ddot{R} f(\xi, 0, \ddot{R}, \dot{\theta})\, d\ddot{R} = \frac{2|A|}{M_{33}}\int_{0}^{\infty} h\, f(\xi, 0, h, \dot{\theta})\, dh \tag{A.J.6}$$

In carrying out the integration involved in the above equation, the following formula will be used:

$$\int_0^{\infty} h\, e^{-h^2 + 2\xi v h}\, dh = \frac{1}{2}\left(1 + \xi v\sqrt{\pi}\, e^{(\xi v)^2} \{1 + \Phi(\xi v)\}\right). \qquad \text{(A.J.7)}$$

Thus, the numerator of Eq. (4.19) becomes

$$\int_{-\infty}^{0} \ddot{R} f(\xi,\, 0,\, \ddot{R},\, \dot{\theta})\, d\ddot{R} = \frac{|\mathbf{A}|^{1/2}}{(2\pi\, M_{33})^{3/2}}\, \xi^2\, e^{-\xi^2 u}\left(1 + \xi v\sqrt{\pi}\, e^{(\xi v)^2}\{1 + \Phi(\xi v)\}\right)$$

$$0 \leq \xi < \infty, \quad -\infty < \dot{\theta} < \infty \qquad \text{(A.J.8)}$$

where $$u = \frac{1}{2|\mathbf{A}|}\{M_{11} + 2\, M_{12}\, \dot{\theta} + (M_{22} - 2\, M_{13})\dot{\theta}^2 - 2\, M_{23}\dot{\theta}^3 + M_{33}\dot{\theta}^4\}$$

$$v = \frac{1}{\{2|\mathbf{A}|\; M_{33}\}^{1/2}}\,(M_{13} + M_{23}\, \dot{\theta} - M_{33}\, \dot{\theta}^2)$$

$$\Phi(x) = \frac{2}{\sqrt{\pi}}\int_0^{x} e^{-t^2}\, dt \qquad \text{(A.J.9)}$$

Next, by applying the integration formula given in Eq. (A.J.7), the denominator of Eq. (4.19) can be written as

$$\int_{-\infty}^{+\infty}\int_0^{\infty}\int_{-\infty}^{0} \ddot{R} f(R,\, 0,\, \ddot{R},\, \dot{\theta})\, d\ddot{R}\, dR\, d\dot{\theta} = \frac{|\mathbf{A}|^{1/2}}{(2\pi\, M_{33})^{3/2}}$$

$$\times \int_{-\infty}^{+\infty}\int_0^{\infty} R^2\, e^{-R^2 u}\left[1 + Rv\sqrt{\pi}\, e^{(Rv)^2}\{1 + \Phi(Rv)\}\right] dR\, d\dot{\theta} \qquad \text{(A.J.10)}$$

Thus, the joint probability density function of $\dot{\theta}$ and R can be obtained from Eqs. (A.J.8) and (A.J.10) as

$$f(\xi,\dot{\theta}) = \frac{\xi^2\, e^{-\xi^2 u}\left[1 + \xi v\sqrt{\pi}\, e^{(\xi v)^2}\{1 + \Phi(\xi v)\}\right]}{\int_{-\infty}^{+\infty}\int_0^{\infty} R^2\, e^{-R^2 u}\left[1 + Rv\sqrt{\pi}\, e^{(Rv)^2}\{1 + \Phi(Rv)\}\right] dR\, d\dot{\theta}} \tag{A.J.11}$$

APPENDIX K PHASE VELOCITY $\dot{\theta}$ AND TIME INTERVAL τ_α

In this Appendix, the phase velocity $\dot{\theta}$ will be expressed approximately in terms of the time interval between two successive peaks of the envelope above a specified level α. For this, the wave profile given in Eq. (4.1) may be written as follows:

$$x(t) = \mathrm{Re} \sum_{n=1}^{\infty} c_n \, e^{i(\omega_n t + \varepsilon_n)}$$

$$= \mathrm{Re}\left\{\rho(t) \cdot e^{i\theta(t)} \cdot e^{i\overline{\omega}t}\right\} \qquad \text{(A.K.1)}$$

where, $\rho(t) \, e^{i\theta(t)} = \sum_{n=1}^{\infty} c_n \, e^{i(\omega - \overline{\omega})t + \varepsilon_n}$

Equation (A.K.1) implies that the wave profile can be expressed as a sinusoidal carrier with the amplitude modulation $\rho(t) \, e^{i\theta(t)}$ varying slowly with time in comparison to the carrier frequency (Longuet-Higgins, 1975).

Let us consider the phase difference $\Delta\theta$ between two successive envelope peaks both of which are above a specified level α. These two peaks are separated in time by an amount approximately equal to the time interval between two envelope crossings at the level α, denoted by τ_α. Then, we may write

$$\Delta\theta = \theta(t + \tau_\alpha) - \theta(t) \qquad \text{(A.K.2)}$$

By expanding the right-hand side of Eq. (A.K.2) into a Taylor series, we have

$$\Delta\theta = \dot{\theta}\,\tau_\alpha + \frac{1}{2}\ddot{\theta}\,\tau_\alpha^2 + \text{-----} \qquad \text{(A.K.3)}$$

Since $\theta(t)$ is a slowly varying function of time, the higher order derivatives of Eq. (A.4.3) may be neglected. On the other hand, the phase difference between two successive envelope peaks (not necessarily above the level α) may be assumed to be approximately equal to 2π. Then, the phase difference between two successive envelope peaks, both of which are above the specified level α, may be written as $2\pi k$. Then, from Eq. (A.K.3), the following relationship can be derived:

$$\dot{\theta}\,\tau_\alpha = 2\pi k \qquad \text{(A.K.4)}$$

The unknown parameter k involved in the above equation can be determined by the iteration method from a consideration of the probability of envelope crossing at the level α. Since the envelope follows the Rayleigh probability distribution, the probability that the envelope R exceeds a specified level α can be evaluated by

$$\Pr\{\text{Envelope exceeds } \alpha\} = \int_\alpha^\infty \frac{R}{m_0} e^{-\frac{R^2}{2m_0}} dR$$

$$= e^{-\frac{\alpha^2}{2m_0}} \qquad \text{(A.K.5)}$$

where, $m_0 = \int_0^\infty S(\omega)\, d\omega.$

This probability is equal to the ratio of the average time duration of the envelope above the level α, denoted by $\overline{\tau}_{\alpha+}$, and the average time interval between two successive envelope crossings, $\overline{\tau}_{\alpha}$. Hence, we have

$$e^{-\frac{\alpha^2}{2m_0}} = \overline{\tau}_{\alpha+}/\overline{\tau}_{\alpha} \qquad \text{(A.K.6)}$$

$\overline{\tau}_{\alpha+}$ can be evaluated from Eq. (4.33) of Chapter 3 of the present study, while $\overline{\tau}_{\alpha}$ can be obtained from Eq. (4.24). The iteration method is applied in determining the k-value. That is, the probability density function $f(\tau_{\alpha}|\alpha)$ given in Eq. (4.24) is obtained by choosing a k-value such that it satisfies the relationship given in Eq. (A.K.6).

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